Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC), REALLY. You are encouraged to use technology to check all of your hand results.

A certain physical system has the following equations of motion and initial conditions:

$$\vec{x}''(t) = A\,\vec{x}(t) + \vec{F}(t),\ \vec{x}(0) = \vec{0},\ \vec{x}'(0) = \vec{0} \text{ where } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ A = \begin{bmatrix} -7 & -6 \\ -1 & -6 \end{bmatrix},\ \vec{F}(t) = \begin{bmatrix} 0 \\ 20\cos(t) \end{bmatrix}$$

a) Rewrite this system of DEs **and** its initial conditions as 6 scalar equations. Solve this system with Maple and write down the solutions for the two unknowns.

b) By hand showing all steps, find the smallest integer component eigenvectors $\vec{b}_1$, $\vec{b}_2$ of the coefficient matrix A produced by the solution algorithm after rescaling of the standard results by positive multiples if necessary, ordered so that the corresponding eigenvalues satisfy $|\lambda_1| < |\lambda_2|$. Evaluate the matrix $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$ and its inverse, and the diagonalized matrix $A_B = B^{-1} A B$. [Use technology to check that your inverse is correct.]

c) What are the slopes m_1, m_2 of the lines through the origin containing the two eigenvectors? On the grid provided, draw in those two lines, labeling them by their corresponding coordinates y_1, y_2 in the positive direction determined by the eigenvectors and then indicate by thicker arrows both eigenvectors $\vec{b}_1$, $\vec{b}_2$, labeled by their symbols. Recall $\vec{x} = B\,\vec{y}$, $\vec{y} = B^{-1}\vec{x}$, where $\vec{y} = \langle y_1, y_2 \rangle$. Also label the x_1, x_2 axes.

d) Find by hand the general solution of the corresponding decoupled system of DEs $\vec{y}'' = A_B\,\vec{y} + B^{-1}\vec{F}$. First write them out in matrix form, then obtain the two equivalent scalar DEs which are its components. Then solve them to find their general solutions using the method of undetermined coefficients, identifying the homogeneous and particular parts of each solution: $y_1 = y_{1h} + y_{1p}$, $y_2 = y_{2h} + y_{2p}$. State your solution in scalar form and box it: $y_1(t) = ...,\ y_2(t) = ...\ .$

e) Then express the general solution for $\vec{x} = B\,\vec{y}$ and impose the initial conditions. Write out and box the scalar solutions: $x_1(t) = ...,\ x_2(t) = ...\ .$ Do they agree with Maple's solution? If not, look for your error. Did you input the equations correctly?

f) Express your (correct) solution as a sum of the two eigenvector modes and the response mode in the form: $\vec{x} = y_{1h}\,\vec{b}_1 + y_{2h}\,\vec{b}_2 + \cos(t)\,\vec{b}_3$ thus identifying the particular solution $\vec{x}_p$ (last term), the response vector coefficient $\vec{b}_3$ and the homogeneous solution $\vec{x}_h$ (first two terms). What are the values of the natural frequencies ω_1 and ω_2 associated with the homogeneous solution? Which frequency is associated with the tandem mode (same signed eigenvector components) and which with the accordian mode (opposite signed eigenvector components)?

g) On the grid provided, draw in the vector $\vec{x}_h(0)$. What are its new coordinates $\langle y_{1h}(0), y_{2h}(0) \rangle$? On your graph, draw in the parallelogram parallel to the new coordinate axes which projects this vector along those axes and identify the sides of the parallelogram on those axes by the number of multiples of the corresponding eigenvector, i.e., $y_{1h}(0)\,\vec{b}_1$ and $y_{2h}(0)\,\vec{b}_2$. Do these seem consistent with your plot?

► **solution**