MAT2500-03/04 10S Test 3 Takehome Answers

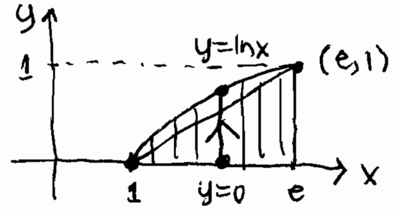
① a) $$A=\int_1^e \underbrace{\int_0^{\ln x} 1\,dy}_{y\big|_{y=0}^{y=\ln x}=\ln x}\,dx=\int_1^e \ln x\,dx$$

Maple

$$= x\ln x - x\Big|_{x=1}^{x=e} = \underbrace{e\underbrace{\ln e}_{=1}-e}_{=0}-[\underbrace{\ln 1}_{=0}-1]$$

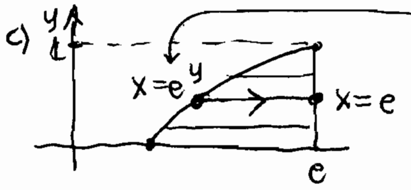
$$=\boxed{1}$$

b) $$\int_{x=1}^{x=e}\int_{y=0}^{y=\ln x} 1\,dy\,dx$$



$$A_\Delta=\frac{1}{2}(e-1)(1)\approx\boxed{0.86}<A$$

c)



$[y=\ln x]$
e

$e^y=e^{\ln x}=x$

d) $$\boxed{\int_0^1\underbrace{\int_{e^y}^{e} 1\,dx}\,dy}=\int_0^1 e-e^y\,dy$$

e) $$x\Big|_{x=e^y}^{x=e}=e-e^y$$

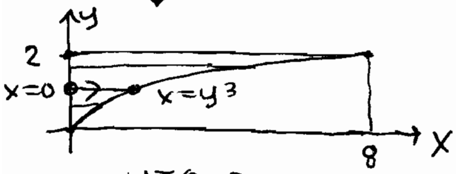
$$= ey-e^y\Big|_0^1$$

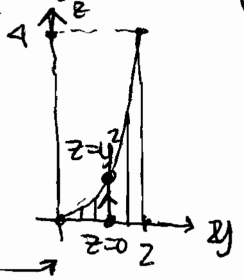
$$= e-e^1-(0-e^0)$$

$$=\boxed{1}$$ agrees with Maple

f) yes!

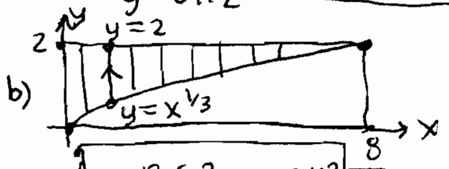
② a) $$\int_{y=0}^{y=2}\int_{x=0}^{x=y3}\int_{z=0}^{z=y2} f\,dz\,dx\,dy$$





y=0..2

b)



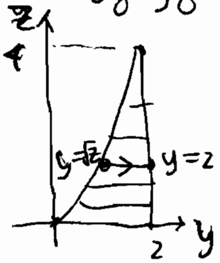
$$\boxed{\underbrace{\int_0^8\int_{x^{1/3}}^{2}}_{\text{new}}\underbrace{\int_0^{y^2}}_{\text{same}} f\,dz\,dy\,dx}$$

② c) original x-y diagram describes x-first innermost integration with x=0..y3

y-z diagram describes outer double integral : z=0..y² while y=0..2.

~~$\int_0^2\int_0^{y2}\int_0^{y3} f\,dx\,dz\,dy$~~

but we want to switch the order:
$z=y^2\rightarrow y=\sqrt{z}$



$$\boxed{\int_0^4\int_{\sqrt{z}}^{2}\int_0^{y^3} f\,dx\,dy\,dz}$$

d) f=1: $$\int_0^2\underbrace{\int_0^{y3}\underbrace{\int_0^{y^2} 1\,dz}_{z\big|_{z=0}^{z=y^2}=y^2}\,dx}_{y^2x\big|_{x=0}^{x=y^3}=y^{2+3}=y^5}\,dy=\int_0^2 y^5\,dy=\frac{y^6}{6}\Big|_0^2$$

$$=\frac{64}{6}=\boxed{\frac{32}{3}}$$

$$\int_0^8\underbrace{\int_{x^{1/3}}^{2}\underbrace{\int_0^{y^2} 1\,dz}_{z\big|_{z=0}^{z=y^2}=y^2}\,dy}_{\frac{y^3}{3}\big|_{y=x^{1/3}}^{y=2}=\frac{8}{3}-\frac{x}{3}}\,dx=\int_0^8\frac{8-x}{3}\,dx$$

$$=\frac{1}{3}\left(8x-\frac{1}{2}x^2\right)\Big|_0^8$$

$$=\frac{1}{3}(64-32)=\boxed{\frac{32}{3}}$$

$$\int_0^4\underbrace{\int_{z^{1/2}}^{2}\underbrace{\int_0^{y3} 1\,dx}_{x\big|_{x=0}^{x=y3}=y^3}\,dy}_{\frac{y^4}{4}\big|_{y=z^{1/2}}^{y=2}=\frac{16}{4}-\frac{z2}{4}}\,dz=\int_0^4\frac{1}{4}(16-z^2)\,dz$$

$$=\frac{1}{4}\left(16z-\frac{z^3}{3}\right)\Big|_{z=0}^{z=4}$$

$$=\frac{1}{4}\left(64-\frac{64}{3}\right)=\boxed{\frac{32}{3}}$$

③ a) $$\underbrace{x^2+y^2}_{r^2}+z^2=4,\quad x^2+y^2+\underbrace{(z-1)^2}_{z^2-2z+1}=1$$

cyl: $$\boxed{r^2+z^2=4,\quad r^2+(z-1)^2=1}$$

sph: $\rho^2=4$, $\boxed{\rho=2}$

$$\rho^2-2(\rho\cos\phi)+1=1$$

$$\underbrace{\rho}_{=0}(\underbrace{\rho-2\cos\phi}) = 0$$

$$\boxed{\rho=2\cos\phi}$$

radial variable solve: $r\ge 0$

$$\boxed{r=\sqrt{4-z^2}},\quad r=\sqrt{1-(z^2-2z+1)}$$

$$\boxed{r=\sqrt{2z-z^2}}$$

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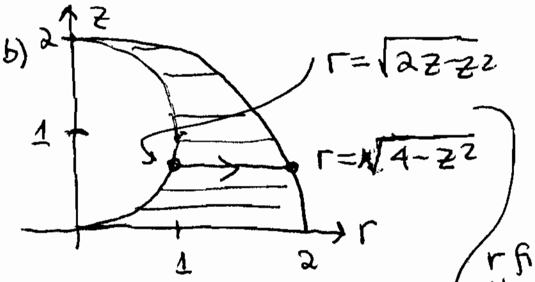
③ b)



r first then z

$z=0..2,\ \theta=0..2\pi$

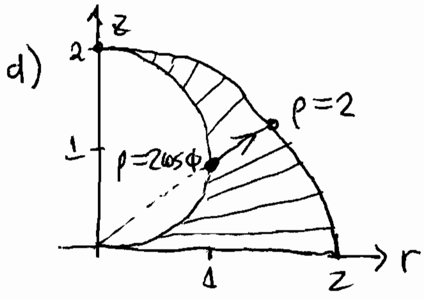
c) $V=\underbrace{\int_0^{2\pi}}_{2\pi}\int_0^2\underbrace{\int_{\sqrt{2z-z^2}}^{\sqrt{4-z^2}}1\cdot r\,dr\,dz\,d\theta}$

$\frac{r^2}{2}\Big|_{r=\sqrt{2z-z^2}}^{r=\sqrt{4-z^2}}=\underbrace{\frac12[-(2z-z^2)+(4-z^2)]}_{=2-z}$

$=2\pi\underbrace{\int_0^2(2-z)\,dz}=\boxed{4\pi}$

$-\frac{z^2}{2}+2z\Big|_0^2=-2+4=2$

d)



$\varphi=0..\frac{\pi}{2},\ \theta=0..2\pi$

e) $V=\underbrace{\int_0^{2\pi}}_{2\pi}\int_0^{\pi/2}\underbrace{\int_{2\cos\phi}^{2}1\ \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta}$

$\frac{\rho^3}{3}\sin\varphi\Big|_{\rho=2\cos\phi}^{\rho=2}$

$\frac13(8-8\cos^3\phi)\sin\phi$

$=2\pi\ \frac83\int_0^{\pi/2}(1-\cos^3\phi)\underbrace{\sin\phi\,d\phi}_{-du}$

$u=\cos\phi$

$-\int(1-u^3)\,du=-u+\frac{u^4}{4}=-\cos\phi+\frac{\cos^4\phi}{4}$

$=2\pi\ \frac83\underbrace{\left(-\cos\phi+\frac{\cos^4\phi}{4}\right)\Big|_0^{\pi/2}}_{+1-\frac14}=2\pi\cdot\frac83\cdot\frac34=\boxed{4\pi}$

f) Agree, also with Maples

g) cylindrical:

$Vz=\underbrace{\int_0^{2\pi}}_{2\pi}\int_0^2\underbrace{\int_{\sqrt{2z-z^2}}^{\sqrt{4-z^2}}z\,r\,dr\,dz\,d\theta}$

$z\frac{r^2}{2}\Big|_{r=\sqrt{2z-z^2}}^{r=\sqrt{4-z^2}}=\frac z2[(4-z^2)-(2z-z^2)]$
$=z(2-z)=2z-z^2$

$=2\pi\underbrace{\int_0^2 2z-z^2\,dz}=\frac{8\pi}{3}$

$z^2-\frac{z^3}{3}\Big|_0^2=4-\frac83=\frac43$

$\bar z=\frac{Vz}{V}=\frac{8\pi/3}{4\pi}=\boxed{\frac23}$

OR spherical:

$Vz=\underbrace{\int_0^{2\pi}}_{2\pi}\int_0^{\pi/2}\underbrace{\int_{2\cos\phi}^{2}(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$

$\underbrace{\frac{\rho^4}{4}\cos\phi\sin\phi\Big|_{\rho=2\cos\phi}^{\rho=2}=\frac14(16-16\cos^4\phi)\cos\phi\sin\phi}$

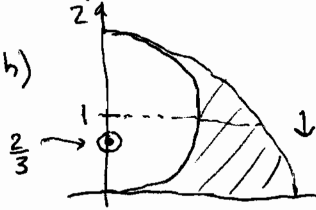
$4\int_0^{\frac{\pi}{2}}\underbrace{(\cos\phi-\cos^5\phi)}_{u-u^5}\underbrace{\sin\phi\,d\phi}_{-du}$

$-4\left(\frac{u^2}{2}-\frac{u^6}{6}\right)=-2\left(\cos^2\phi-\frac{\cos^6\phi}{3}\right)$

$=2\pi\left(-2\left(\cos^2\phi-\frac{\cos^6\phi}{3}\right)\right)\Big|_0^{\pi/2}$

$=2\pi(+2)\left(1-\frac13\right)=8\pi/3\qquad \bar z=\frac{Vz}{V}=\frac{8\pi/3}{4\pi}=\boxed{\frac23}$

h)

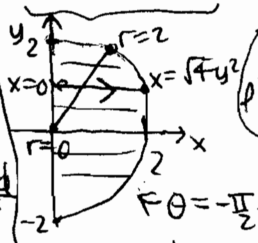


↓ more volume below halfway point, shifts centroid down a bit, seems right.

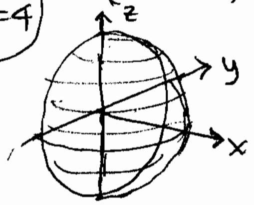
④ a) $\int_{y=-2}^{y=2}\int_{x=0}^{x=\sqrt{4-y^2}}\int_{z=-\sqrt{4-x^2-y^2}}^{z=\sqrt{4-x^2-y^2}}y^2\underbrace{\sqrt{x^2+y^2+z^2}}_{\rho}\ \underbrace{dz\,dx\,dy}_{dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$

$y^2 \leftarrow (\rho\sin\phi\sin\theta)^2$

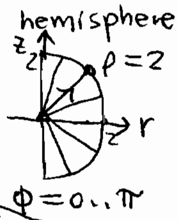
$z^2=4-x^2-y^2$
$\rho^2=x^2+y^2+z^2=4$
$\rho=2$



$\theta=-\frac{\pi}{2}..\frac{\pi}{2}$
$r=0..2$



"eastern hemisphere"



$\phi=0..\pi$

$\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\pi}\int_0^2\underbrace{\rho^2\rho\cdot\rho^2\sin^2\phi\sin^2\theta\sin\phi}_{\rho^5\sin^3\phi\ \sin^2\theta}\,d\rho\,d\phi\,d\theta}$

$=\left(\frac{\pi}{2}\right)\left(\frac43\right)\left(\frac{32}{3}\right)=\boxed{\frac{64}{9}\pi}\approx\underbrace{22.34021433}$

b) ↗

c) $=\underbrace{\int_{-\frac{\pi}{2}}^{\pi/2}\sin^2\theta\,d\theta}\ \underbrace{\int_0^{\pi}\sin^3\phi\,d\phi}\ \underbrace{\int_0^2\rho^5\,d\rho}$ (Factors)

$-\frac12\sin\theta\cos\theta+\frac{1}{2}\Big|_{-\pi/2}^{\pi/2}=\frac12(\pi)=\pi/2$

$-\frac13\sin^2\phi\cos\phi-\frac23\cos\phi\Big|_0^{\pi}=+\frac23+\frac23=4/3$

$\rho^6/6\Big|_0^2=32/3$

d) agrees with Maple evaluation of this & original Cartesian integral