

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad).

1. A coupled second order system of DEs with initial conditions is the following:

$$x_1'' = -5x_1 + 2x_2 - \cos(2t), x_2'' = x_1 - 4x_2 + 2\cos(2t), x_1(0) = -2, x_1'(0) = 4, x_2(0) = 4, x_2'(0) = 1.$$

a) Rewrite this system of DEs and its initial conditions in explicit matrix form for the (column matrix!) vector variable $\vec{x} = \langle x_1, x_2 \rangle$, identifying the coefficient matrix A and forcing vector function \vec{F} , namely: $\vec{x}'' = A\vec{x} + \vec{F}$, $\vec{x}(0) = \vec{x}_0, \vec{x}'(0) = \vec{v}_0$.

b) Use the Maple right click menu find the eigenvalues and eigenvectors of this coefficient matrix A . Write them down identifying the eigenvalues λ_1, λ_2 (ordered so that $|\lambda_1| < |\lambda_2|$ so we all agree on the ordering of the new coordinates) and the corresponding matrix of eigenvectors $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$.

c) Evaluate the new components $\vec{y}(0) = B^{-1}\vec{x}(0)$ and $\vec{y}'(0) = B^{-1}\vec{x}'(0)$ of the vectors $\vec{x}(0)$ and $\vec{x}'(0)$.

d) On the graph paper on the reverse side, draw in the two eigenvectors as arrows (label them) and their associated new coordinate lines y_1, y_2 , and the initial vectors $\vec{x}(0)$ and $\vec{x}'(0)$ (label them!). Graphically determine their new components with respect to the new basis using your diagram, drawing in the parallelogram associated with their projections along the new basis vectors down to the new coordinate lines for each vector. Is your drawing consistent with your matrix calculated components? Explain.

e) Introduce the 4 component vector variable $\vec{X} = \langle x_1, x_2, x_3, x_4 \rangle = \langle x_1, x_2, x_1', x_2' \rangle$, and re-express the DEs and initial conditions in the first order matrix form

$$\vec{X}' = \mathcal{M}\vec{X} + \vec{f}, \vec{X}(0) = \vec{X}_0.$$

► solution

① a)
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'' = \begin{bmatrix} -5x_1 + 2x_2 - \cos 2t \\ x_1 - 4x_2 + 2\cos 2t \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 2 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\cos 2t \\ 2\cos 2t \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -5 & 2 \\ 1 & -4 \end{bmatrix}$$

b) Maple: $\begin{bmatrix} -6 \\ -3 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ reverse order
 $1-3 < 1-6$

$$\lambda_1 = -3, \lambda_2 = -6$$

$$B = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \langle \vec{b}_1 | \vec{b}_2 \rangle$$

$$c) B^{-1} = \frac{1}{1+2} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2+8 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \vec{y}(0)$$

$$\begin{bmatrix} y_1'(0) \\ y_2'(0) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4+2 \\ -4+1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \vec{y}'(0)$$

d) see reverse

e)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_1' \\ x_2' \end{bmatrix} \rightarrow \begin{bmatrix} x_1' \\ x_2' \\ x_3'' \\ x_4'' \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_3'' \\ x_4'' \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -5x_1 + 2x_2 - \cos 2t \\ x_1 - 4x_2 + 2\cos 2t \end{bmatrix}$$
 } def } DE

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 2 & 0 & 0 \\ 1 & -4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\cos 2t \\ 2\cos 2t \end{bmatrix}, \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 4 \\ 1 \end{bmatrix}$$

► gridline default plot window

yes. The graphical linear combinations agree with the matrix result.

