MAT2705-02/06 10F Quiz 4 Print Name (Last, First) ________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC).

The relative birth rate $\beta > 0$ and the relative death rate $\delta > 0$ for a population model are defined by the equation
$\frac{dP}{dt} = (\beta - \delta)\, P$.

Consider a prolific breed of rabbits whose birth and death rates, β and δ, are each proportional to the rabbit population $P = P(t)$, with $\beta > \delta$.

a) Show that $P(t) = \frac{P_0}{1 - k P_0 t}$, k constant. Note that $P(t) \to \infty$ as $t \to \frac{1}{(k P_0)}$. This is doomsday.

b) What sign must k have and why?

c) Suppose that $P_0 = 6$ and that there are nine rabbits after ten months. When does doomsday occur?

▶ solution

distinct! different variables, different behavior

a) $\beta \propto P,\ \delta \propto P \rightarrow \beta = k_1 P,\ \delta = k_2 P$

$$\frac{dP}{dt} = (\beta - \delta) P = (k_1 P - k_2 P) P = \underbrace{(k_1 - k_2)}_{\equiv k} P^2 \quad \text{separable}$$

$$\int P^{-2}\, dP = \int k\, dt$$

$$-P^{-1} = kt + C_1$$

$$P^{-1} = -kt \underbrace{- C_1}_{C} = C - kt$$

$$P = \frac{1}{C - kt} \quad \text{(gen soln)}$$

$$P(0) = P_0 = \frac{1}{C - 0} = \frac{1}{C} \rightarrow C = \frac{1}{P_0}$$

$$\boxed{P = \frac{1}{\frac{1}{P_0} - kt} \left(\frac{P_0}{P_0}\right) = \frac{P_0}{1 - k P_0 t}} \quad \text{(IVP soln)}$$

$$\left(1 - k P_0 t \approx 0 \rightarrow t = \frac{1}{k P_0} \right)$$

b) $\beta - \delta = k_1 P - k_2 P = \underbrace{(k_1 - k_2)}_{\therefore\, = k > 0} \underbrace{P}_{\geq 0} > 0$

so $\boxed{k > 0}$ net growth

c) $P_0 = 6,\ P(10) = 9 = \frac{6}{1 - 60k}$

$1 - 60k = \frac{6}{9} = \frac{2}{3}$

$1 - \frac{2}{3} = 60k \rightarrow k = \frac{1}{60}\left(\frac{1}{3}\right) = \frac{1}{180}$

$$P = \frac{6}{1 - \frac{6}{180} t} = \frac{6}{1 - \frac{t}{30}} = \frac{180}{30 - t}$$

$\boxed{t = 30 \text{ months}}$ (time units are months)