Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC), REALLY. You are encouraged to use technology to check all of your hand results.

A certain coupled pair of oscillators has the following equations of motion and initial conditions:
$x_1" = -12\ x_1 + 2\ x_2$, $x_2" = 16\ x_1 - 8\ x_2 + 70\ \sin(3\ t)$, $x_1(0) = 2$, $x_2(0) = 2$, $x_1'(0) = 0$, $x_2'(0) = 0$.

a) Rewrite this system of DEs *and* its initial conditions in explicit matrix form for the (column matrix) vector variable $\vec{x} = \langle x_1, x_2 \rangle$, identifying the coefficient matrix A.

b) By hand showing all steps, find the smallest integer component eigenvectors $\vec{b}_1$, $\vec{b}_2$ of the coefficient matrix A produced by the solution algorithm after rescaling of the standard results by positive multiples, ordered so that the corresponding eigenvalues satisfy $|\lambda_1| < |\lambda_2|$. Evaluate the matrix $B = \langle \vec{b}_1 | \vec{b}_2 \rangle$ and its inverse. [Use technology to check that your inverse is correct.]

c) What are the slopes M_1, M_2 of the lines through the origin containing the two eigenvectors? On the grid provided, draw in those two lines, labeling them by their corresponding coordinates y_1, y_2 in the positive direction determined by the eigenvectors and then indicate by thicker arrows both eigenvectors $\vec{b}_1$, $\vec{b}_2$, labeled by their symbols. Recall $\vec{x} = B\vec{y}$, $\vec{y} = B^{-1}\vec{x}$, where $\vec{y} = \langle y_1, y_2 \rangle$. Also label the x_1, x_2 axes.

d) Now by hand solve the initial value problem stated above before part a).
First write out the new decoupled equations $\vec{y}" = A_B\vec{y} + B^{-1}\vec{f}$ first in vector form, then the two equivalent scalar equations. Then solve them to find their general solution. State it and box it: $y_1(t) = ...$, $y_2(t) = ...$.
Then express the general solution for $\vec{x}$ and impose the initial conditions. Write out the scalar solution:
$x_1(t) = ...$, $x_2(t) = ...$. Does it agree with Maple's solution? If not, write down Maple's solution and look for your error. Did you input the equations correctly?

e) Express your (correct) solution as a sum of the two eigenvector modes and the response mode in the form:
$\vec{x} = y_{1h}\vec{b}_1 + y_{2h}\vec{b}_2 + \sin(3\ t)\vec{b}_3$ thus identifying the particular solution (last term), the response vector coefficient $\vec{b}_3$ and the homogeneous solution $\vec{x}_h$ (first two terms). Also write down your solutions $y_{1h}(t) = ...$, $y_{2h}(t) = ...$.

f) On the grid provided, draw in the vector $\vec{x}(0) = \langle 2, 2 \rangle$ .What are its new coordinates $(y_1(0), y_2(0))$? On your graph, draw in the parallelogram parallel to the new coordinate axes which projects this vector along those axes and identify the sides of the parallelogram on those axes by the number of multiples of the corresponding eigenvector, i. e., $y_1(0)\vec{b}_1$ and $y_2(0)\vec{b}_2$. Do these seem consistent with your plot?

g) Evaluate the amplitudes (A_1, A_2) and phase shifts (δ_1, δ_2) of $y_{1h}(t)$ and $y_{2h}(t)$. Make sure each computation is accompanied by a diagram showing the appropriate quadrant needed to get the phase shift angle correct. Give their exact values and their numerical values, with the phase shift numerical values in degrees to the nearest degree.
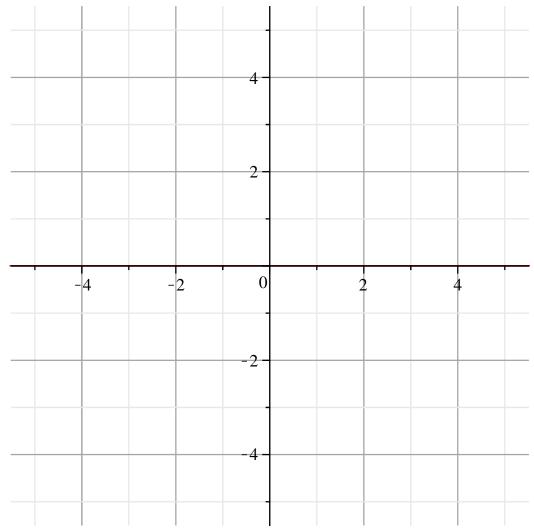
▶ **solution**

gridline default plot window

> $plot(0, view = [-5.5 .. 5.5, -5.5 .. 5.5], gridlines = true)$

The actual motion is pretty crazy due to the driving force