# Math 2705-02/06 10F TakeHome Test 3 Answers

1 a) $[1000\,x'' + 2000\,x' + 10\,000\,x = F]\,/1000$

$$\boxed{x'' + 2x' + 10x = F/1000}$$

$k_0 = 2$, $\tau_0 = 1/2 = 0.5$ $\qquad \omega_0^2 = 10$, $\omega_0 = \sqrt{10} \approx 3.162$ $\qquad Q = \omega_0\tau_0 = \tfrac{1}{2}\sqrt{10} \approx 1.581$

$T_0 = \dfrac{2\pi}{\sqrt{10}} \approx 1.987$

b) $x = e^{kt} \to \underbrace{(r^2+2r+10)}_{=0} e^{rt} = 0$

$r = \dfrac{-2 \pm \sqrt{4-40}}{2} = -1 \pm 3i$

$e^{rt} = e^{-t}e^{\pm 3it} \xrightarrow[\text{basis}]{\text{real}} e^{-t}\cos 3t,\ e^{-t}\sin 3t$

$$\boxed{x = e^{-t}(c_1\cos 3t + c_2 \sin 3t)}$$

$\tau_1 = 1$, $\omega_1 = 3$, $T_1 = \frac{2\pi}{3} \approx 2.094$

c) $F/m = \sin 4t$

$10\ [x_p = c_3\cos 4t + c_4 \sin 4t$

$2\ [x_p' = -4c_3 \sin 4t + 4c_4\cos 4t$

$1\ [x_p'' = -16c_3\cos 4t - 16 c_4 \sin 4t$

$x_p'' + 2x_p' + 10x_p = [(10-16)c_3 + 8c_4]\cos 4t + [-8c_3 + (10-16)c_4]\sin 4t = \sin 4t$

$-6c_3 + 8c_4 = 0$
$-8c_3 - 6c_4 = 1$ $\qquad \begin{bmatrix} -6 & 8 \\ -8 & -6 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \dfrac{1}{36+64}\begin{bmatrix} -6 & -8 \\ 8 & -6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\dfrac{1}{100}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = -\dfrac{1}{50}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$x_p = -\frac{1}{50}(4\cos 4t + 3\sin 4t)$

$x_h = e^{-t}(c_1\cos 3t + c_2 \sin 3t)$

$x = e^{-t}(c_1\cos 3t + c_2\sin 3t) - \frac{1}{50}(4\cos 4t + 3\sin 4t)$

$x' = -e^{-t}(c_1\cos 3t + c_2 \sin 3t) - \frac{1}{50}(-16\sin 4t + 12\cos 4t)$
$\quad + e^{-t}(-3c_1\sin 3t + 3c_2\cos 3t)$

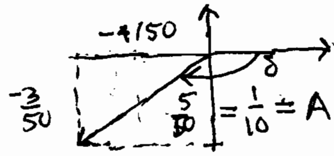
$x(0) = c_1 - 4/50 = 0 \to c_1 = 4/50 = 2/25$

$x'(0) = -c_1 + 3c_2 - 12/50 = 0 \to c_2 = \frac{1}{3}\left(\frac{12}{50} + \frac{4}{50}\right) = \frac{8}{75}$

$$\boxed{x = \underbrace{\tfrac{2}{75}e^{-t}(3\cos 3t + 4\sin 3t)}_{\text{transient.}} - \underbrace{\tfrac{1}{50}(4\cos 4t + 3\sin 4t)}_{\text{steady state}}}$$

$\left(-\frac{4}{50}, -\frac{3}{50}\right)$

c) (continued)




$A = \frac{1}{10}$, $\delta = -\pi + \arctan\frac{3}{4} \approx -143.130°$
$\approx -0.398$ cycles

shifted left, ahead in time

relative to sin: $\delta - \pi/4 \approx -0.648$ cycle $\sim 2/3$ cycle ahead (to left of sin)

looks right in diagram, see plot

d) $F/m = A_0\omega^2 \sin(\omega t)$

$10\ [x_p = c_3\cos\omega t + c_4 \sin\omega t$

$2\ [x_p' = -c_3\omega\sin\omega t + c_4\omega\cos\omega t$

$1\ [x_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4 \sin\omega t$

$x_p'' + 2x_p' + 10x_p'' = [(10-\omega^2)c_3 + 2\omega c_4]\cos\omega t = A_0\omega^2 \sin\omega t$
$\qquad + [-2\omega c_3 + (10-\omega^2)c_4]\sin\omega t$

$(10-\omega^2)c_3 + 2\omega c_4 = 0$
$-2\omega c_3 + (10-\omega^2)c_4 = A_0\omega^2$ $\qquad \begin{bmatrix} 10-\omega^2 & 2\omega \\ -2\omega & 10-\omega^2 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ A_0\omega^2 \end{bmatrix}$

$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \dfrac{1}{(10-\omega^2)^2 + 4\omega^2}\begin{bmatrix} 10-\omega^2 & -2\omega \\ 2\omega & 10-\omega^2 \end{bmatrix}\begin{bmatrix} 0 \\ A_0\omega^2 \end{bmatrix}$
$\quad = \dfrac{A_0\omega^2}{(10-\omega^2)^2 + 4\omega^2}\begin{bmatrix} -2\omega \\ 10-\omega^2 \end{bmatrix}$

$$\boxed{x_p = \frac{A_0\omega^2}{(10-\omega^2)^2 + 4\omega^2}\left[-2\omega\cos\omega t + (10-\omega^2)\sin\omega t\right]}$$

oops, no e). Cut to make test shorter.

f) $A(\omega) = \dfrac{\sqrt{(-2\omega)^2 + (10-\omega^2)^2}\ A_0\omega^2}{(10-\omega^2)^2 + 4\omega^2}$

$\quad = \dfrac{A_0\omega^2}{\sqrt{(10-\omega^2)^2 + 4\omega^2}}$

continued on page 3

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① h) $A(4) = \frac{4\cdot 4^2}{\sqrt{(10-16)^2+4(16)}} = \frac{64}{10} = \boxed{6.4 \text{ inches}}$

peak: $A(\omega_p) \approx 1.664\, A_0 \approx \boxed{6.656 \text{ inches}}$

i) $\frac{A(\omega)}{A_0} = \frac{\omega^2}{\sqrt{(10-\omega^2)^2+4\omega^2}} = \frac{\omega^2}{\sqrt{\omega^4-20\omega^2+100+4\omega^2}}$

$= \frac{\omega^2}{\omega^2\sqrt{1-\underbrace{\frac{16}{\omega^2}}_{\to 0}+\underbrace{\frac{100}{\omega^4}}_{\to 0}}} \to 1$ as $\omega\to\infty$

same amplitude!

$x_p = \frac{A_0\omega^2}{(10-\omega^2)^2+4\omega^2}[-2\omega\cos\omega t + (10-\omega^2)\sin\omega t]$

leading behavior: $\frac{A_0}{\omega^2}$

only this term can car

$\sim A_0\left[-\underbrace{\frac{2}{\omega}}_{\to 0}\cos\omega t - \frac{\omega^2}{\omega^2}\sin\omega t\right]$

$\to -A_0\sin\omega t$, 180° out of phase →

equal but opposite displacement to ground floor so the second floor remains fixed in space while the ground floor oscillates under it, unable to respond to the high frequency oscillation: $x_{space} = x_p + x_0 \to 0$

② a) $x_1' = 0x_1 + x_2$, $x_2' = -10x_1 - 2x_2$, $x_1(0) = -1$, $x_2(0) = 4$

b) $A = \begin{bmatrix}0 & 1\\ -10 & -2\end{bmatrix} \to 0 = |A-\lambda I| = \begin{vmatrix}-\lambda & 1\\ -10 & -2-\lambda\end{vmatrix} =$

$= \lambda(\lambda+2)+10 = \lambda^2+2\lambda+10 \to$

$\lambda = -1 \pm 3i$ (problem ①!)

$A-(-1+3i) = \begin{bmatrix}1-3i & 1\\ -10 & -1-3i\end{bmatrix} \xrightarrow[\times(\frac{1}{10})]{RREF} \begin{bmatrix}1 & \frac{1+3i}{10}\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$x_2 = t$, $x_1 = -(1+3i)/10\; t$, $\begin{bmatrix}x_1\\ x_2\end{bmatrix} = t\underbrace{\begin{bmatrix}-(1+3i)/10\\ 1\end{bmatrix}}_{\vec b_1 = \overline{\vec b_2}}$

$B = \begin{bmatrix}(-1-3i)/10 & (-1+3i)/10\\ 1 & 1\end{bmatrix}$

let $\vec x = B\vec y$, $\vec y = B^{-1}\vec x$, $B^{-1}AB = \begin{bmatrix}-1+3i & 0\\ 0 & -1-3i\end{bmatrix}$

$(\vec x' = A\vec x) \to B^{-1}[(B\vec y)' = A(B\vec y)]$

$\vec y' = B^{-1}AB\,\vec y \Rightarrow y_1' = (-1+3i)y_1$, $y_1 = e^{(-1+3i)t}C_1$

$y_2' = (-1-3i)y_2$, $y_2 = e^{(-1-3i)t}C_2$

$\vec x = B\vec y = y_1\vec b_1 + y_2\vec b_2 = C_1 e^{(-1+3i)t}\vec b_1 +$ c.c. (if $C_2 = \overline{C_1}$)

general soln but complex form

② b) (continued)

complex soln: $e^{(-1+3i)t}\begin{bmatrix}(-1-3i)/10\\ 1\end{bmatrix}$

$= e^{-t}(\cos 3t + i\sin 3t)\begin{bmatrix}-\frac{1}{10}(1+3i)\\ 1\end{bmatrix}$

$= e^{-t}\begin{bmatrix}-\frac{1}{10}[\cos 3t - 3\sin 3t + i(\sin 3t + 3\cos 3t)]\\ \cos 3t + i\sin 3t\end{bmatrix}$

$= \underbrace{e^{-t}\begin{bmatrix}-\frac{1}{10}(\cos 3t - 3\sin 3t)\\ \cos 3t\end{bmatrix} + i\, e^{-t}\begin{bmatrix}-\frac{1}{10}(\sin 3t + 3\cos 3t)\\ \sin 3t\end{bmatrix}}_{\text{real solns}}$

$$\boxed{\begin{bmatrix}x_1\\ x_2\end{bmatrix} = c_1 e^{-t}\begin{bmatrix}-\frac{1}{10}(\cos 3t - 3\sin 3t)\\ \cos 3t\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}-\frac{1}{10}(\sin 3t + 3\cos 3t)\\ \sin 3t\end{bmatrix}}$$

general soln (explicitly real)

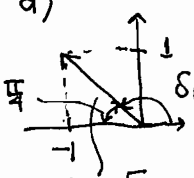
$\begin{bmatrix}-1\\ 4\end{bmatrix} = \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = c_1\begin{bmatrix}-1/10\\ 1\end{bmatrix} + c_2\begin{bmatrix}-3/10\\ 0\end{bmatrix} = \begin{bmatrix}-\frac{1}{10}c_1 - \frac{3}{10}c_2\\ c_1\end{bmatrix}$

$\to c_1 = 4$, $c_2 = \frac{10}{3}[1 - \frac{1}{10}(4)] = 2$

$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = e^{-t}\begin{bmatrix}-\frac{4}{10}(\cos 3t - 3\sin 3t) + -\frac{2}{10}(\sin 3t + 3\cos 3t)\\ 4\cos 3t \qquad\qquad +2\sin 3t\end{bmatrix}$

$= e^{-t}\begin{bmatrix}-\cos 3t + \sin 3t\\ 4\cos 3t + 2\sin 3t\end{bmatrix}$

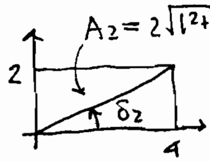
d)




$\delta_1 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$A_1 = \sqrt{2}$

$A_2 = 2\sqrt{1^2+2^2} = 2\sqrt{5}$




$\delta_2 = \arctan\frac{1}{2}$

$x_1 = \sqrt{2}\, e^{-t}\cos(3t - 3\pi/4)$

$x_2 = 2\sqrt{5}\, e^{-t}\cos(3t - \arctan 1/2)$

$x_1$ envelope: $\pm\sqrt{2}\, e^{-t}$

$x_2$ envelope: $\pm 2\sqrt{5}\, e^{-t}$

$\left(\frac{A_1}{A_2} = \frac{\sqrt{2}}{2\sqrt{5}} = \frac{1}{\sqrt{10}} \approx 0.316\right)$

$\frac{\delta_1 - \delta_2}{2\pi} = \frac{3\pi/4 - \arctan 1/2}{2\pi} \approx \boxed{0.301}$

conclusion: $x_1$ has about 31% of the amplitude of $x_2$ and lags behind $x_2$ in time by about 30% of a cycle.

see plot.

③ a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & -2 \\ 5 & -7 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$0 = |A - \lambda I| = \begin{vmatrix} -\lambda & -2 \\ 5 & -7-\lambda \end{vmatrix} = \lambda(\lambda+7) + 10 = \lambda^2 + 7\lambda + 10$

$\lambda = \frac{-7 \pm \sqrt{49-40}}{2} = \frac{-7 \pm 3}{2} = -5, -2$

$\lambda_1 = -2, \ \lambda_2 = -5, \quad \lambda_1 > \lambda_2$

$\lambda_1$: $A + 2I = \begin{bmatrix} 2 & -2 \\ 5 & -5 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t, \ x_1 = x_2 = t, \ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda_2$: $A + 5I = \begin{bmatrix} 5 & -2 \\ 5 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -2/5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t, \ x_1 = \frac{2}{5} x_2 = \frac{2}{5} t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t/5 \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 2/5 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$B = \begin{bmatrix} 1 & 2/5 \\ 1 & 1 \end{bmatrix} = \langle \vec{b}_1 | \vec{b}_2 \rangle$

b) $\vec{x} = B\vec{y} \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2/5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\vec{y} = B^{-1} \vec{x} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{1 - \frac{2}{5}} \begin{bmatrix} 1 & -2/5 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{5}{3} \begin{bmatrix} -1 - 4/5 \\ 1 + 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$\uparrow \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

c) see plot

d) soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-5t} \begin{bmatrix} 2/5 \\ 1 \end{bmatrix}$

$\left( \begin{matrix} c_1 = y_1(0) = -3 \\ c_2 = y_2(0) = 5 \end{matrix} \right)$

$= \begin{bmatrix} -3e^{-2t} + 2e^{-5t} \\ -3e^{-2t} + 5e^{-5t} \end{bmatrix}$

$\tau_1 = \frac{1}{2} > \tau_2 = \frac{1}{5}$

minimum:

$x_2 = -3e^{-2t} + 5e^{-5t}$

$0 = x_2' = 6e^{-2t} - 25e^{-5t} \,]\, e^{5t}$

$6e^{3t} - 25 = 0$

$e^{3t} = 25/6, \ \boxed{t = \frac{1}{3} \ln \frac{25}{6} \approx 0.476}$

$\hookrightarrow x_2 = -3e^{-\frac{2}{3} \ln \frac{25}{6}} + 5e^{-\frac{5}{3} \ln \frac{25}{6}}$

$= -3\left(\frac{25}{6}\right)^{-2/3} + 5\left(\frac{25}{6}\right)^{-5/3}$

$= -3\left(\frac{6}{25}\right)^{2/3} + 5\left(\frac{6}{25}\right)^{5/3}$

$= \left(\frac{6}{25}\right)^{2/3} \left[-3 + 5 \cdot \frac{6}{25}\right] = \boxed{-\frac{9}{5}\left(\frac{6}{25}\right)^{2/3} \approx -0.695}$

$= -\frac{9}{5} \frac{(6)^{2/3}}{(25)^{2/3}} = -\frac{9}{5^{7/3}} \cdot \frac{5^{2/3}}{\ldots} = -\frac{9}{125} 5^{2/3} 6^{2/3}$ Maple answer

① f) $(10-\omega^2)^2 + 4\omega^2 = 100 - 20\omega^2 + \omega^4 + 4\omega^2 = 100 - 16\omega^2 + \omega^4$

$\frac{A(\omega)}{A_0} = \frac{\omega^2}{(100 - 16\omega^2 + \omega^4)^{1/2}}$ use quotient rule

$0 = \frac{A'(\omega)}{A_0} = \frac{(\ldots)^{1/2} 2\omega - \omega^2 \left(\frac{1}{2}\right)(\ldots)^{-1/2}(-32\omega + 4\omega^3)}{(\ldots)^1}$

add fractions: $\frac{a z^{1/2} - b/z^{1/2}}{z} = \frac{a z^1 - b}{z^{3/2}}$

$= \frac{2\omega}{(\ldots)^{3/2}} \left[ \underbrace{(100 - 16\omega^2 + \omega^4)^1 - \omega^2(-8 + \omega^2)} \right]$

$100 - 16\omega^2 + \omega^4 + 8\omega^2 - \omega^4 = 0$

$100 - 8\omega^2 = 0 \quad \omega^2 = \frac{25}{2}$

$\boxed{\omega_p = \frac{5}{\sqrt{2}} = \frac{5}{2}\sqrt{2} \approx 3.536} \gtrsim \omega_0 \approx 3.162$

$\frac{A(\omega_p)}{A_0} = \frac{25/2}{(100 - 16(25/2) + 25^2/4)^{1/2}} = \frac{25}{(400 - 800 + 625)^{1/2}} = \frac{25}{\sqrt{225}} = \frac{25}{\sqrt{9 \cdot 25}} = \boxed{\frac{5}{3} \approx 1.667} \leftrightarrow Q \approx 1.581$

comparable

$A(4) = \frac{(A_0 4^2) = 1}{\sqrt{(10-16)^2 + 4(16)}} = \frac{1}{\sqrt{64+36}} = \frac{1}{\sqrt{100}} = \boxed{\frac{1}{10}}$ ✓ yes, agrees

① g) see plot

$\frac{A(\omega_0)}{A_0} = \frac{10}{(100 - 160 + 100)^{1/2}} = \frac{10}{40^{1/2}} = \frac{\sqrt{10}}{2} \approx 1.581$

(for plot)