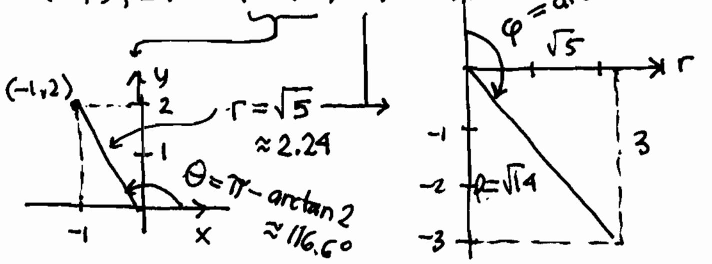
MAT 2500-03/04 09S Final Exam Answers

① $(x,y,z) = (-1,2,-3)$



$= \pi - \arctan\left(\frac{\sqrt5}{3}\right) = \frac{\pi}{2} + \arctan\left(\frac{3}{\sqrt5}\right)$

$\approx 143.3°$

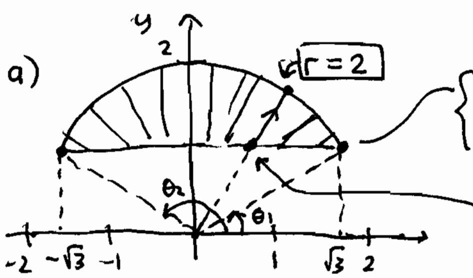
a bit more than 45° beyond 90° ✓

a) cylindrical: $(r,\theta,z) = (\sqrt5, \pi - \arctan 2, -3)$

a bit short of 90° plus 45° : $\approx 116.6°$ ✓

b) spherical: $(\rho,\theta,\phi) = (\sqrt{14}, \text{(same)}, \pi - \arctan\frac{\sqrt5}{3})$
$\approx 143.3°$

② a)



intersection:
$\begin{cases} x^2+y^2 = 4 \to x^2 = 3 \\ y = 1 \end{cases}$ $x = \pm\sqrt3$

$r\sin\theta = 1 \to \boxed{r = \csc\theta}$

$\boxed{r = \csc\theta .. 2}$

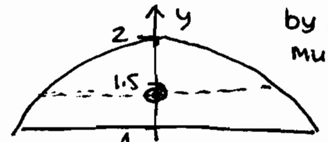
$\theta_1 = \arctan(1/\sqrt3) = \pi/6 = 30°$
$\theta_2 = \pi - \theta_1 = 5\pi/6 = 150°$
$\boxed{\theta = \theta_1 .. \theta_2 = \frac{\pi}{6} .. \frac{5\pi}{6}}$

b) $A_y = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\csc\theta}^{2} \underbrace{(r\sin\theta)}_{y}\, \underbrace{r\,dr\,d\theta}_{dA}$

$= \int_{\frac{\pi}{6}}^{5\pi/6}\int_{\csc\theta}^{2} r^2\sin\theta\, dr\, d\theta \overset{\text{maple}}{=} \boxed{2\sqrt3}$

$A = \int_{\frac{\pi}{6}}^{5\pi/6}\int_{\csc\theta}^{2} r\,dr\,d\theta \overset{\text{maple}}{=} \boxed{\frac{4\pi}{3} - \sqrt3}$

$\boxed{\bar{y} = \frac{2\sqrt3}{\frac{4\pi}{3} - \sqrt3} \approx 1.410}$



by symmetry must lie on y axis.

seems reasonable to me!

c) The contribution of C_2 is very small compared to C_1. The vector field makes a small acute angle with $-\hat{T}$ on C_1 so it makes a big negative contribution. Result should be negative.

d) C_1: $\vec r(t) = \langle 2\cos t, 2\sin t\rangle$ $t = \frac{\pi}{6} .. \frac{5\pi}{6}$
$\vec r\,'(t) = \langle -2\sin t, 2\cos t\rangle$
$\vec F(\vec r(t)) = \langle (2\sin t)^3, -(2\cos t)^3\rangle = \langle 8\sin^3 t, -8\cos^3 t\rangle$
$F(\vec r(t))\cdot \vec r\,'(t) = -16\sin^4 t - 16\cos^4 t$

$\int_{C_1} \vec F\cdot d\vec r = \int_{\frac{\pi}{6}}^{5\pi/6} -16\sin^4 t - 16\cos^4 t\, dt = \sqrt3 - 8\pi \approx -23.4$

C_2: $\vec r(t) = \langle t, 1\rangle$ $t = -\sqrt3 .. \sqrt3$
$\vec r\,'(t) = \langle 1, 0\rangle$
$\vec F(\vec r(t)) = \langle 1^3, -t^3\rangle$
$\vec F(\vec r(t))\cdot \vec r\,'(t) = 1$
$\int_{C_2} \vec F\cdot d\vec r = \int_{-\sqrt3}^{\sqrt3} 1\, dt = 2\sqrt3 \approx 3.46$
$\oint_C \vec F\cdot d\vec r = 2\sqrt3 + (\sqrt3 - 8\pi)$
$= \boxed{3\sqrt3 - 8\pi \approx -19.937}$
<0 as predicted.

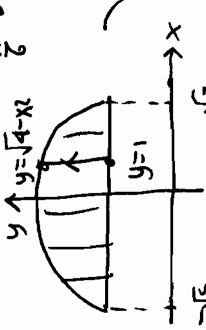
alternate C_1: $y = \sqrt{4-x^2}$, $x = -\sqrt3 .. \sqrt3 = t \to \vec r(t) = \langle t, \sqrt{4-x^2}\rangle$ reverse direction
$\vec r\,'(t) = \langle 1, -t/\sqrt{4-t^2}\rangle$, $\vec F(\vec r(t)) = \langle (4-t^2)^{3/2}, -t^3\rangle \to \vec F(\vec r(t)) = (4-t^2)^{3/2} + t^4/\sqrt{4-t^2} \to -\int_{-\sqrt3}^{\sqrt3} \%_0\, dt = \ldots = \sqrt3 - 8\pi$

e) $\frac{\partial}{\partial x}(\underbrace{-x^3}_{F_2}) - \frac{\partial}{\partial y}(\underbrace{y^3}_{F_1}) = -3x^2 - 3y^2 = -3(x^2+y^2) = -3r^2$

$\int_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\csc\theta}^{2} (-3r^2)\, r\, dr\, d\theta \boxed{= -8\pi + 3\sqrt3 \approx -19.937}$ ✓ same result

f) $\int_{-\sqrt3}^{\sqrt3}\int_{1}^{\sqrt{4-x^2}} -3(x^2+y^2)\, dy\, dx \overset{\text{maple}}{=}$ same result after simplification.

OR $\int_1^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} -3(x^2+y^2)\, dx\, dy$



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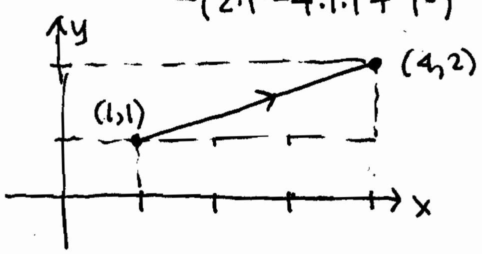
(3) a) $\vec{F} = \langle 4x-4y, -4x+2y \rangle$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(-4x+2y) - \frac{\partial}{\partial y}(4x-4y)$$
$$= -4+4 = 0 \checkmark$$

b) $\vec{F} = \vec{\nabla} f$:

$$\int \left[\frac{\partial f}{\partial x} = 4x-4y\right] dx \rightarrow f = \int 4x-4y\,dx$$
$$= 2x^2-4xy+C(y)$$
$$\frac{\partial f}{\partial y} = -4x+2y$$
$$0-4x + C'(y) = -4x+2y$$
$$C'(y) = 2y$$
$$C(y) = y^2+k$$
$$\boxed{f = 2x^2-4xy+y^2+k} \quad k=0 \text{ simplest choice}$$

c) $$\int_C \vec{F}\cdot d\vec{r} = f(4,2)-f(1,1)$$
$$= \begin{matrix} 2\cdot 4^2-4\cdot 4\cdot 2+2^2 = 4 \\ -(2\cdot 1^2-4\cdot 1\cdot 1+1^2) \quad -(-1) \end{matrix} = \boxed{5}$$

d) 

$$\vec{r}(t) = \langle 1,1\rangle + t(\langle 4,2\rangle - \langle 1,1\rangle)$$
$$= \langle 1,1\rangle + t\langle 3,1\rangle$$
$$= \langle 1+3t, 1+t\rangle \quad t=0..1$$
$$\vec{r}\,'(t) = \langle 3,1\rangle$$
$$\vec{F}(\vec{r}(t)) = \langle 4(1+3t)-4(1+t), -4(1+3t)+2(1+t)\rangle$$
$$= \langle 8t, -2-10t\rangle$$
$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = 3(8t)+1(-2-10t) = 14t-2$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 14t-2\,dt = 7t^2-2t\Big|_0^1 = 7-2 = \boxed{5} \checkmark$$

OR: $y-1 = \frac{1}{3}(x-1) \rightarrow y = \frac{x+2}{3} \quad x = 1..4 = t$

$$\vec{r}(t) = \langle t, \tfrac{t+2}{3}\rangle \quad \vec{r}\,'(t) = \langle 1, 1/3\rangle$$
$$\vec{F}(\vec{r}(t)) = \langle 4t-4(\tfrac{t+2}{3}), -4t+2(\tfrac{t+2}{3})\rangle$$
$$= \tfrac{1}{3}\langle 8t-8, -10t+4\rangle$$
$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \tfrac{1}{3}(8t-8+\tfrac{1}{3}(-10t+4))$$
$$= (14t-20)/9$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_1^4 \frac{14t-20}{9}\,dt$$
$$= \frac{7t^2-20}{9}\Big|_1^4 = \dots 5$$

OR $x = 3y-2,\ y = 1..2 = t \rightarrow \vec{r}(t) = \langle 3t-2, t\rangle$, $\vec{r}\,'(t) = \langle 3,1\rangle$

$$\vec{F}(\vec{r}(t)) = \dots = \langle 8t-8, -10t+8\rangle$$
$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \dots = 14t-16$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_1^2 14t-16\,dt = \dots = 5$$