MAT2705-02/06 09F FINAL EXAM ANSWERS (1)

a) $x_1 = \frac{1}{2}\cos t - \frac{4}{5}\cos 2t + \frac{3}{10}\cos 3t$

$x_2 = 3\cos t - \frac{12}{5}\cos 2t - \frac{3}{5}\cos 3t$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4/5\cos 2t + 3/10\cos 3t + 1/2\cos t \\ -12/5\cos 2t - 3/5\cos 3t + 3\cos t \end{bmatrix}$$

$$= \cos 2t \underbrace{\begin{bmatrix} -4/5 \\ -12/5 \end{bmatrix}}_{=\vec{x}} + \cos 3t \underbrace{\begin{bmatrix} 3/10 \\ -3/5 \end{bmatrix}}_{=\vec{y}} + \cos t \underbrace{\begin{bmatrix} 1/2 \\ 3 \end{bmatrix}}_{=\vec{z}}$$

$\omega_1 = 2$ same sign: tandem $\qquad \omega_2 = 3$ opp sign: accordion $\qquad \omega = 1$ same sign tandem

$T_1 = \frac{2\pi}{2} = \pi, \quad T_2 = \frac{2\pi}{3}, \quad T = \frac{2\pi}{1} = 2\pi$

$T = 2T_1 = 3T_2$ is the common period

amplitude ratios

$$\frac{4/5}{12/5} = \frac{1}{3}, \quad \frac{3/10}{3/5} = \frac{1}{2}, \quad \frac{1/2}{3} = \frac{1}{6}$$

$\omega_1 = 2, \omega_2 = 3$ natural modes of system

$\omega = 1$ driving → response mode

b)
$$\underbrace{\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix}}_{\vec{x}''} = \begin{bmatrix} -7x_1 + x_2 \\ 6x_1 - 6x_2 + 12\cos t \end{bmatrix} = \underbrace{\begin{bmatrix} -7 & 1 \\ 6 & -6 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ 12\cos t \end{bmatrix}}_{\vec{F}}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

c)
$$0 = |A - \lambda I| = \begin{vmatrix} -7-\lambda & 1 \\ 6 & -6-\lambda \end{vmatrix} = (-7-\lambda)(-6-\lambda) - 6$$
$$= (\lambda+7)(\lambda+6) - 6 = \lambda^2 + 13\lambda + 42 - 6 = \lambda^2 + 13\lambda + 36 = (\lambda+4)(\lambda+9) = 0$$

$\lambda = -4, -9$

$\lambda = -4$: $A + 4I = \begin{bmatrix} -7+4 & 1 \\ 6 & -6+4 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$

L F

$\begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 - \frac{1}{3}x_2 = 0, \ x_1 = t/3$

$x_2 = t$ $\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/3 \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda = -9$: $A + 9I = \begin{bmatrix} -7+9 & 1 \\ 6 & -6+9 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$

L F

$\begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 + \frac{1}{2}x_2 = 0, \ x_1 = -1/2\, t$

$x_2 = t$ $\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1/2\, t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

c) continued

$$B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\frac{1}{3} + \frac{1}{2}} \begin{bmatrix} 1 & 1/2 \\ -1 & 1/3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{3}{5} \\ -\frac{6}{5} & \frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 6 & 3 \\ -6 & 2 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix} = A_D$$

d) $\vec{b}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$m_1 = \frac{1}{1/3} = 3 \qquad m_2 = \frac{1}{-1/2} = -2$

$x_2 = 3x_1, \qquad x_2 = -2x_1$

(eigenspace equations)

$$B^{-1}\vec{f} = \frac{1}{5}\begin{bmatrix} 6 & 3 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 12\cos t \end{bmatrix} = \begin{bmatrix} \frac{36}{5}\cos t \\ \frac{24}{5}\cos t \end{bmatrix}$$

$$\begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 36/5\cos t \\ 24/5\cos t \end{bmatrix}$$

$$\left.\begin{aligned} y_1'' &= -4y_1 + \tfrac{36}{5}\cos t \\ y_2'' &= -9y_2 + \tfrac{24}{5}\cos t \end{aligned}\right\} \text{decoupled equations}$$

$y_1'' + 4y_1 = \frac{36}{5}\cos t$

$y_h = c_1\cos 2t + c_2\sin 2t$

$4[y_{1p} = c_5\cos t]$

$0[y_{1p}' = -c_5\sin t]$

$1[y_{1p}'' = -c_5\cos t]$

$y_{1p}'' + 4y_{1p} = (4-1)c_5\cos t = \frac{36}{5}\cos t$

$\therefore 3c_5 = \frac{36}{5}, \ c_5 = \frac{12}{5}, \ y_{1p} = \frac{12}{5}\cos t$

$y_2'' + 9y_2 = \frac{24}{5}\cos t$

$y_h = c_3\cos 3t + c_4\sin 3t$

$9[y_{2p} = c_6\cos t]$

$0[y_{2p}' = -c_6\sin t]$

$1[y_{2p}'' = -c_6\cos t]$

$y_{2p}'' + 9y_{2p} = (9-1)c_6\cos t = 24/5\cos t$

$\therefore 8c_6 = \frac{24}{5}, \ c_6 = \frac{3}{5}, \ y_{2p} = \frac{3}{5}\cos t$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1\cos 2t + c_2\sin 2t + 12/5\cos t \\ c_3\cos 3t + c_4\sin 3t + 3/5\cos t \end{bmatrix} \text{ gen soln.}$$

e) continued

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t + 12/5 \cos t \\ c_3 \cos 3t + c_4 \sin 3t + 3/5 \cos t \end{bmatrix}$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2c_1 \sin 2t + 2c_2 \cos 2t - 12/5 \sin t \\ -3c_3 \sin 3t + 3c_4 \cos 3t - 3/5 \sin t \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 + 12/5 \\ c_3 + 3/5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} c_1 + 12/5 \\ c_3 + 3/5 \end{bmatrix} = B^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} c_1 = -12/5 \\ c_3 = -3/5 \end{matrix}$$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = B^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} c_2 = 0 \\ c_4 = 0 \end{matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -12/5 \cos 2t + 12/5 \cos t \\ -3/5 \cos 3t + 3/5 \cos t \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}\left(-\frac{12}{5}\cos 2t + \frac{12}{5}\cos t\right) - \frac{1}{2}\left(-\frac{3}{5}\cos 3t + \frac{3}{5}\cos t\right) \\ \left(-\frac{12}{5}\cos 2t + \frac{12}{5}\cos t\right) + \left(-\frac{3}{5}\cos 3t + \frac{3}{5}\cos t\right) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{5}\cos 2t + \frac{3}{10}\cos 3t + \overbrace{\left(\frac{4}{5} - \frac{3}{10}\right)}^{=1/2} \cos t \\ -\frac{12}{5}\cos 2t - \frac{3}{5}\cos 3t + \underbrace{\left(\frac{12}{5} + \frac{3}{5}\right)}_{=3} \cos t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5}\cos 2t + \frac{3}{10}\cos 3t + \frac{1}{2}\cos t \\ -\frac{12}{5}\cos 2t - \frac{3}{5}\cos 3t + 3\cos t \end{bmatrix}$$

$$= \cos 2t \begin{bmatrix} -4/5 \\ -12/5 \end{bmatrix} + \cos 3t \begin{bmatrix} 3/10 \\ -3/5 \end{bmatrix} + \cos t \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$$

final solution, same as Maple.

f) $\begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} = \begin{bmatrix} -7 & 1 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ → homogeneous soln found above

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cos 2t + c_2 \sin 2t \\ c_3 \cos 3t + c_4 \sin 3t \end{bmatrix} \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2c_1 \sin 2t + 2c_2 \cos 2t \\ -3c_3 \sin 3t + 3c_4 \cos 3t \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & 3 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3+9 \\ -3+6 \end{bmatrix} = \begin{bmatrix} 12/5 \\ 3/5 \end{bmatrix}$$ new coordinates of $\vec{z} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} c_2 = 0 \\ c_4 = 0 \end{matrix}$$ as above.

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 12/5 \cos 2t \\ 3/5 \cos 3t \end{bmatrix} = \frac{12}{5}\cos 2t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} + \frac{3}{5}\cos 3t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{5}\cos 2t - \frac{3}{10}\cos 3t \\ \frac{12}{5}\cos 2t + \frac{3}{5}\cos 3t \end{bmatrix}}$$

$\vec{z} = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix} = \frac{12}{5}\vec{b}_1 + \frac{3}{5}\vec{b}_2 = 2.4\vec{b}_1 + 0.6\vec{b}_2$ looks perfect on graph

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