# MAT 2705-02/06 09F Take Home Test 3 Answers (1)

① a) $y'' + \underbrace{4}_{k_0}y' + \underbrace{40}_{\omega_0^2}y = F(t)$, $y(0) = 0 = y'(0)$.

$1 \to k_0$

$k_0 = 4$, $\tau_0 = k_0^{-1} = 1/4 = 0.25$, $\omega_0 = \sqrt{40} = 2\sqrt{10} \approx 6.325$

$Q = \omega_0\tau_0 = \frac{1}{4}2\sqrt{10} = \frac{\sqrt{10}}{2} \approx 1.581$

$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{2\sqrt{10}} = \frac{\pi}{\sqrt{10}} \approx 0.993$

b) $F(t) = 0$, homogeneous solution, $y \sim e^{rt}$

$\to \underbrace{(r^2+4r+40)}_{=0}e^{rt} = 0$

$=0 \to r = \frac{-4 \pm \sqrt{4^2 - 4\cdot 40}}{2} = -2 \pm \sqrt{-9}$ 2 mental omission

$= -2 \pm 6i$  $(3 \to 6 = \omega)$

$e^{rt} = e^{(-2\pm 6i)t} = e^{-2t}(\cos 6t \pm i \sin 6t)$

→ real basis soln space: $\{e^{-2t}\cos 6t, e^{-2t}\sin 6t\}$

$$\boxed{y = e^{-2t}(c_1\cos 6t + c_2 \sin 6t)}$$ gen soln

$k_1 = 2$, $\tau_1 = \frac{1}{2} = 0.5$   $\omega_1 = 6$

$T_1 = \frac{2\pi}{6} \approx$ ~~2.094~~ $1.047$

c) $F(t) = 1$, $\to y_p = c_3 \to$ backsub

$0 + 0 + 40c_3 = 1 \to c_3 = \frac{1}{40} = 0.025$

$y_h = y$ (part b).

$$\boxed{y = y_h + y_p = e^{-2t}(c_1\cos 6t + c_2\sin 6t) + \frac{1}{40}}$$

$5\tau_1 = 2.5$ so plot $t = 0..2.5$

since $T_1 \approx 1$, will see a bit more than 2 oscillation periods

$$\boxed{\lim_{t\to\infty} y = \frac{1}{40} = 0.025}$$ → horizontal asymptote! $y = 0.025$

see plot at end. IVP soln at end (oops)

d) $F(t) = \sin 6t$

$40\,[y_p = c_3\cos 6t + c_4 \sin 6t]$

$4\,[y_p' = -6c_3\sin 6t + 6c_4\cos 6t]$

$1\,[y_p'' = -36c_3\cos 6t - 36c_4\sin 6t]$

$y_p'' + 4y_p' + 40y_p = [(40-36)c_3 + 4(6)c_4]\cos 6t$
$+ [-4(6)c_3 + (40-36)]\sin 6t$
$= \sin 6t$

$4c_3 + 24c_4 = 0$
$-24c_3 + 4c_4 = 1$
$4\begin{bmatrix}1 & 6\\ -6 & 1\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$

① d) continued: $\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 6\\ -6 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}$

$= \frac{1}{4}\frac{1}{37}\begin{bmatrix}1 & -6\\ 6 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{4\cdot 37}\begin{bmatrix}-6\\ 1\end{bmatrix}$

$y_p = \frac{1}{4\cdot 37}[-6\cos 6t + \sin 6t]$

$y_h = y$ (part b)   1/148

gen soln $$\boxed{y = e^{-2t}(c_1\cos 6t + c_2\sin 6t) + \frac{1}{4\cdot 37}(-6\cos 6t + \sin 6t)}$$

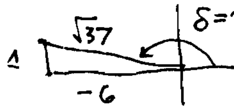
$y' = -2e^{-2t}(c_1\cos 3t + c_2\sin 3t) + \frac{1}{4\cdot 37}(36\sin 6t + 6\cos 6t)$
$+ e^{-2t}(-6c_1\sin 3t + 6c_2\cos 3t)$

$0 = y(0) = c_1 - \frac{6}{4\cdot 37} \to c_1 = \frac{3}{2\cdot 37} = \frac{3}{74}$

$0 = y'(0) = -2c_1 + 6c_2 + \frac{6}{4\cdot 37}$

$c_2 = \frac{1}{6}\left[2\left(\frac{3}{2\cdot 37}\right) - \frac{3}{2\cdot 37}\right] = \frac{1}{6}\frac{3}{2\cdot 37} = \frac{1}{2\cdot 74} = \frac{1}{148}$

IVP soln
$$\boxed{y = \frac{1}{148}e^{-2t}(\underbrace{6\cos 6t}_{3/74} + \sin 6t) + \underbrace{\frac{1}{148}(-6\cos 6t}_{=-3/74} + \sin 6t)}$$




$\delta = \pi - \arctan\left(\frac{1}{6}\right) \approx 170.5° \approx 0.474$ cycles

$A = \frac{1}{148}\sqrt{37} = \frac{1}{4\sqrt{37}} \approx 0.041$

see plot at end. phase difference with $\sin 6t$: $0.474 - 0.250 = 0.224$ cycles
so about 1/5 cycle behind $\sin 6t$
see graph

e) $F(t) = \sin \omega t$

$40\,[y_p = c_3\cos\omega t + c_4\sin\omega t]$

$4\,[y_p' = -\omega c_3\sin\omega t + \omega c_4\cos\omega t]$

$1\,[y_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t]$

$y_p'' + 4y_p' + 40y_p = [(40-\omega^2)c_3 + 4\omega c_4]\cos\omega t$
$+ [-4\omega c_3 + (40-\omega^2)c_4]\sin\omega t$
$= \sin\omega t$

$(40-\omega^2)c_3 + 4\omega c_4 = 0$
$-4\omega c_3 + (40-\omega^2)c_4 = 1$
$\begin{bmatrix}40-\omega^2 & 4\omega\\ -4\omega & 40-\omega^2\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$

$\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}40-\omega^2 & 4\omega\\ -4\omega & 40-\omega^2\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix} = \frac{1}{(40-\omega^2)^2+16\omega^2}\begin{bmatrix}40-\omega^2 & -4\omega\\ 4\omega & 40-\omega^2\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}$

$= \frac{1}{(40-\omega^2)^2+16\omega^2}\begin{bmatrix}-4\omega\\ 40-\omega^2\end{bmatrix}$

$$\boxed{y_p = \frac{-4\omega\cos\omega t + (40-\omega^2)\sin\omega t}{(40-\omega^2)^2 + 16\omega^2} = y_{ss}}$$

MAT2705-02/06 09F Take Home Test 3 Answers (2)

1 f) $A = \dfrac{\sqrt{(-4\omega)^2 + (40-\omega^2)^2}}{(40-\omega^2)^2 + 16\omega^2} = \left((40-\omega^2)^2 + 16\omega^2\right)^{-1/2}$

$A'(\omega) = -\frac{1}{2}(\cdots)^{-3/2}\left(\underline{2(40-\omega^2)(-2\omega) + 32\omega}\right) = 0$

$0 = 4\omega[\omega^2 - 40 + 8] = 4\omega(\omega^2 - 32)$

$\omega = 0, \pm\sqrt{32} \xrightarrow[\omega>0]{} \boxed{\omega_p = \sqrt{32} = 4\sqrt{2} \approx 5.657}$

$A(\omega_p) = [\underbrace{(40-32)^2}_{64=16\cdot 4} + 16\cdot 32]^{-1/2} = [16(4+32)]^{-1/2}$

$= (4^2\cdot 6^2)^{-1/2} = \frac{1}{4\cdot 6} = \boxed{\frac{1}{24} \approx 0.0417}$

$A(0) = \frac{1}{(40^2)^{1/2}} = \frac{1}{40}$

$\frac{A(\omega_p)}{A(0)} = \frac{1/24}{1/40} = \frac{40}{24} = \boxed{\frac{5}{3} \approx 1.67}$

$Q \approx 1.58$ not so different – "comparable"

$A(6) = [(40-36)^2 + 16\cdot 36]^{-1/2}$

$= (16 + 16\cdot 36)^{-1/2} = 16^{-1/2}\, 37^{-1/2} = \frac{1}{4\sqrt{37}}$ ✓

agrees with previous result.

g) see plots at end

2. a) $x_1' = -5x_1 + 3x_2,\ x_2' = -6x_1 + x_2,$

$x_1(0) = 1,\ x_2(0) = 2.$

b) $A = \begin{bmatrix} -5 & 3 \\ -6 & 1 \end{bmatrix}$ $|A - \lambda I| = \begin{vmatrix} -5-\lambda & 3 \\ -6 & 1-\lambda \end{vmatrix}$

$= (\lambda+5)(\lambda-1) + 18 = \lambda^2 + 4\lambda - 5 + 18$

$= \lambda^2 + 4\lambda + 13 = 0$

$\lambda = \frac{-4 \pm \sqrt{16 - 4\cdot 13}}{2} = -2 \pm \sqrt{-9} = -2 \pm 3i$

$\lambda = -2+3i$:

$$A - \lambda I = \begin{bmatrix} -5+2-3i & 3 \\ -6 & 1+2-3i \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & -\frac{1}{2}+\frac{i}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = t \quad x_1 = \left(\frac{1}{2} - \frac{i}{2}\right)t \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2} - \frac{i}{2}\right)t \\ t \end{bmatrix} = t \underbrace{\begin{bmatrix} \frac{1-i}{2} \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda = -2-3i,\ \vec{b}_2 = \begin{bmatrix} \frac{1+i}{2} \\ 1 \end{bmatrix}$

2 b) continued:

$B = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ 1 & 1 \end{bmatrix}$ $A_D = B^{-1}AB = \begin{bmatrix} -2+3i & 0 \\ 0 & -2-3i \end{bmatrix}$

$\vec{x}' = A\vec{x} \leftarrow \vec{x} = B\vec{y},\ \vec{y} = B^{-1}\vec{x}$

$B^{-1}[(B\vec{y})' = A(B\vec{y})] \to \vec{y}' = A_D\vec{y}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -2+3i & 0 \\ 0 & -2+3i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (-2+3i)y_1 \\ (-2-3i)y_2 \end{bmatrix}$

$y_1' = (-2+3i)y_1,\ y_1 = c_1 e^{(-2+3i)t}$

$y_2' = (-2-3i)y_2 \quad y_2 = c_2 e^{(-2-3i)t}$

$\vec{x} = B\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2 =$

$= \underbrace{c_1 e^{(-2+3i)t}\begin{bmatrix} (1-i)/2 \\ 1 \end{bmatrix}}_{//} + c_2 e^{(-2-3i)t}\begin{bmatrix} (1+i)/2 \\ 1 \end{bmatrix}$

$= e^{-2t}(\cos 3t + i\sin 3t)\begin{bmatrix} (1-i)/2 \\ 1 \end{bmatrix}$

$= e^{-2t}\begin{bmatrix} \frac{1}{2}\cos 3t + \frac{1}{2}\sin 3t + \frac{i}{2}\sin 3t - \frac{i}{2}\cos 3t \\ \cos 3t + i\sin 3t \end{bmatrix}$

$= \underbrace{e^{-2t}\begin{bmatrix} \frac{1}{2}\cos 3t + \frac{1}{2}\sin 3t \\ \cos 3t \end{bmatrix}}_{\vec{X}_1} + i\underbrace{e^{-2t}\begin{bmatrix} \frac{1}{2}\sin 3t - \frac{1}{2}\cos 3t \\ \sin 3t \end{bmatrix}}_{\vec{X}_2}$

new basis of soln space.

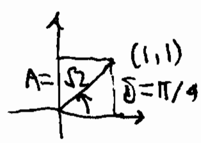
$$\boxed{\begin{aligned} \vec{x} &= c_1\vec{X}_1 + c_2\vec{X}_2 \qquad \text{gen soln} \\ &= c_1 e^{-2t}\begin{bmatrix} \frac{1}{2}\cos 3t + \frac{1}{2}\sin 3t \\ \cos 3t \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} \frac{1}{2}\sin 3t - \frac{1}{2}\cos 3t \\ \sin 3t \end{bmatrix} \end{aligned}}$$

c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}c_1 - \frac{1}{2}c_2 \\ c_1 \end{bmatrix}$

$\therefore c_1 = 2,\ 1 = \frac{1}{2}(2) - \frac{1}{2}c_2 \to c_2 = 0$

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos 3t + \sin 3t \\ 2\cos 3t \end{bmatrix}}$$

d) $= e^{-2t}\begin{bmatrix} \sqrt{2}\cos(3t - \pi/4) \\ 2\cos(3t) \end{bmatrix}$




envelopes:

$x_1: \pm\sqrt{2}e^{-2t}$

$x_2: \pm 2e^{-2t}$

e) see plots at end.

2 f) $x_1 = x_2$ :

$$e^{-2t}(\cos 3t + \sin 3t) = 2e^{-2t}\cos 3t$$

$$\cos 3t + \sin 3t = 2\cos 3t$$

$$\sin 3t = \cos 3t \to 3t = \pi/4$$

$$t = \frac{\pi}{12} \approx 0.262$$

$$x_1 = x_2 = 2e^{-2\pi/12}\cos\frac{3\pi}{12} = 2e^{-\pi/6}\cos\frac{\pi}{4}$$

$$= \frac{2}{\sqrt{2}}e^{-\pi/6} = \underbrace{\sqrt{2}}\, e^{-\pi/6} \approx 0.838$$

$x_1$ upper envelope curve value.

see plots at end.

3 a) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} -11 & 3 \\ -2 & -4 \end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$0 = |A - \lambda I| = \begin{vmatrix} -11-\lambda & 3 \\ -2 & -4-\lambda \end{vmatrix} = (\lambda+4)(\lambda+11)+6$$

$$= \lambda^2 + 15\lambda + 50 \to \lambda = \frac{-15 \pm \sqrt{15^2 - 4\cdot 50}}{2}$$

$$\overset{\text{maple}}{=} \cdots = -5, -10$$

$\lambda = -5$: $A + 5I = \begin{bmatrix} -6 & 3 \\ -2 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t$, $x_1 = \frac{1}{2}t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda = -10$: $A + 10I = \begin{bmatrix} -1 & 3 \\ -2 & 6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_2 = t$, $x_1 = 3t$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$$\boxed{\begin{array}{c} \lambda = -5, -10 \\ B = \langle \vec{b}_1 | \vec{b}_2 \rangle = \begin{bmatrix} 1/2 & 3 \\ 1 & 1 \end{bmatrix} \end{array}} \quad A_D = \begin{bmatrix} -5 & 0 \\ 0 & -10 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \to \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B^{-1} = \frac{1}{\underbrace{\frac{1}{2} - 3}_{-\frac{5}{2}}}\begin{bmatrix} 1 & -3 \\ -1 & 1/2 \end{bmatrix} = -\frac{2}{5}\begin{bmatrix} 1 & -3 \\ -1 & 1/2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}$$

3 b) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \to \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 & 6 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 20 \\ -5 \end{bmatrix}$

$$= \boxed{\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}$$

c) The direction field arrows line up along the straight lines containing the eigenvectors so yes, they agree. (see plots at end)

d) The general soln (not requested) is just

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{-5t}\vec{b}_1 + c_2 e^{-10t}\vec{b}_2$$

with $\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = B\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ so $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

from our calculation above so:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 4e^{-5t}\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} - e^{-10t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-5t} - 3e^{-10t} \\ 4e^{-5t} - e^{-10t} \end{bmatrix}$$

you could have just gotten this directly from Maple.

local max of $x_1 = 2e^{-5t} - 3e^{-10t}$ :

$$[x_1' = -10e^{-5t} + 30e^{-10t} = 0]\, e^{10t}$$

$$-10e^{5t} + 30 = 0, \; e^{5t} = 3, \; \boxed{t = \tfrac{1}{5}\ln 3 \approx 0.220}$$

$$x_1(\tfrac{1}{5}\ln 3) = 2e^{-\ln 3} - 3e^{-2\ln 3}$$

$$= 2(3)^{-1} - 3(3)^{-2} = \frac{2}{3} - \frac{3}{9} = \boxed{\frac{1}{3} \approx 0.333}$$

This point clearly agrees with the graph.
The larger characteristic time is

$\tau_1 = 1/5$, $5\tau_1 = 1$.

The plot window $t = 0..2$ shows both variables merging with axis pixels by $t = 1.3$.

The next pages are PDF output from Maple of the 7 requested plots. The Maple originals look much better.

1 c) continued (oops)

$$y = e^{-2t}(c_1\cos 6t + c_2\sin 6t) + 1/40$$

$$y' = -2e^{-2t}(c_1\cos 6t + c_2\sin 6t) + e^{-2t}(-6c_1\sin 6t + 6\cos 6t)$$

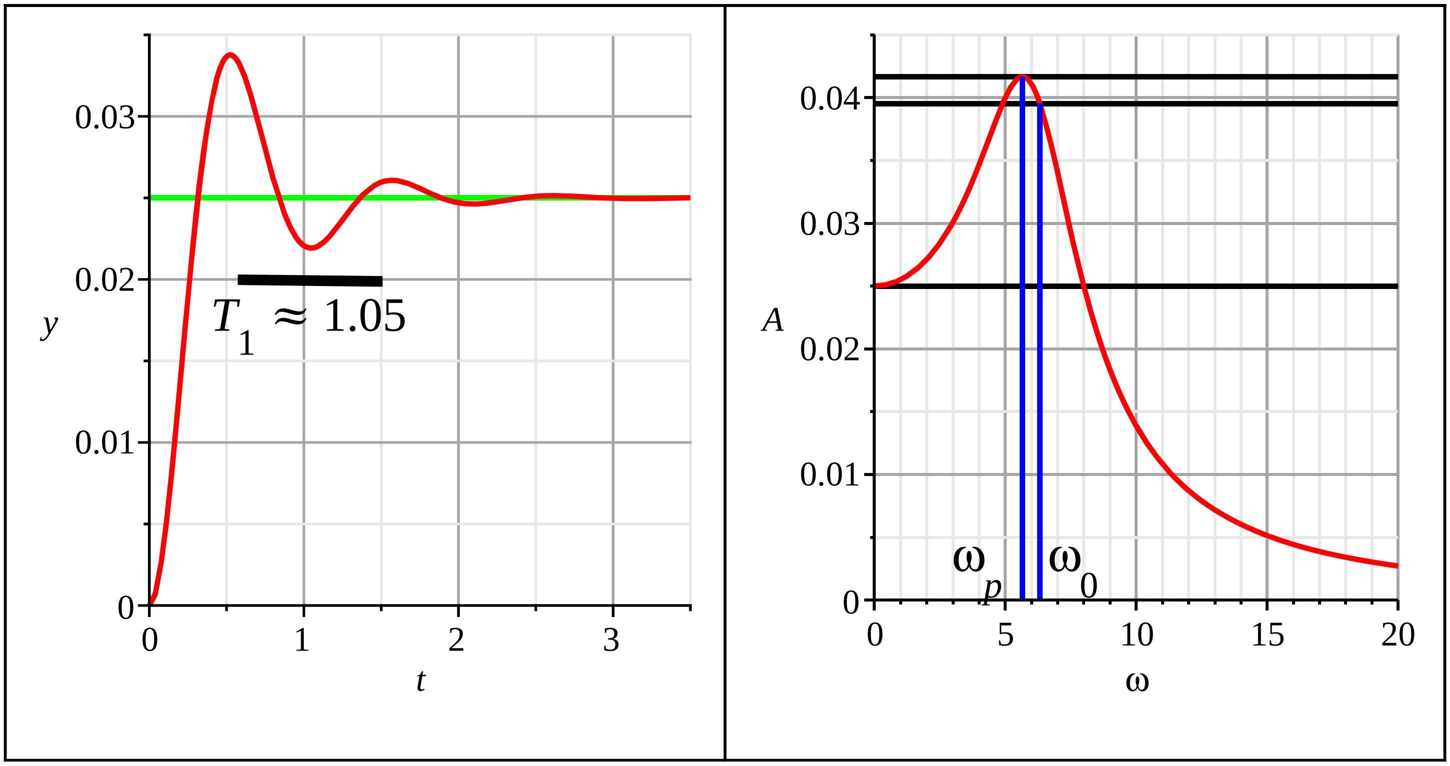
$$y(0) = c_1 + \tfrac{1}{40} = 0 \to c_1 = -1/40$$

$$y'(0) = -2c_1 + 6c_2 = 0 \to c_2 = \tfrac{1}{3}c_1 = -\tfrac{1}{120}$$

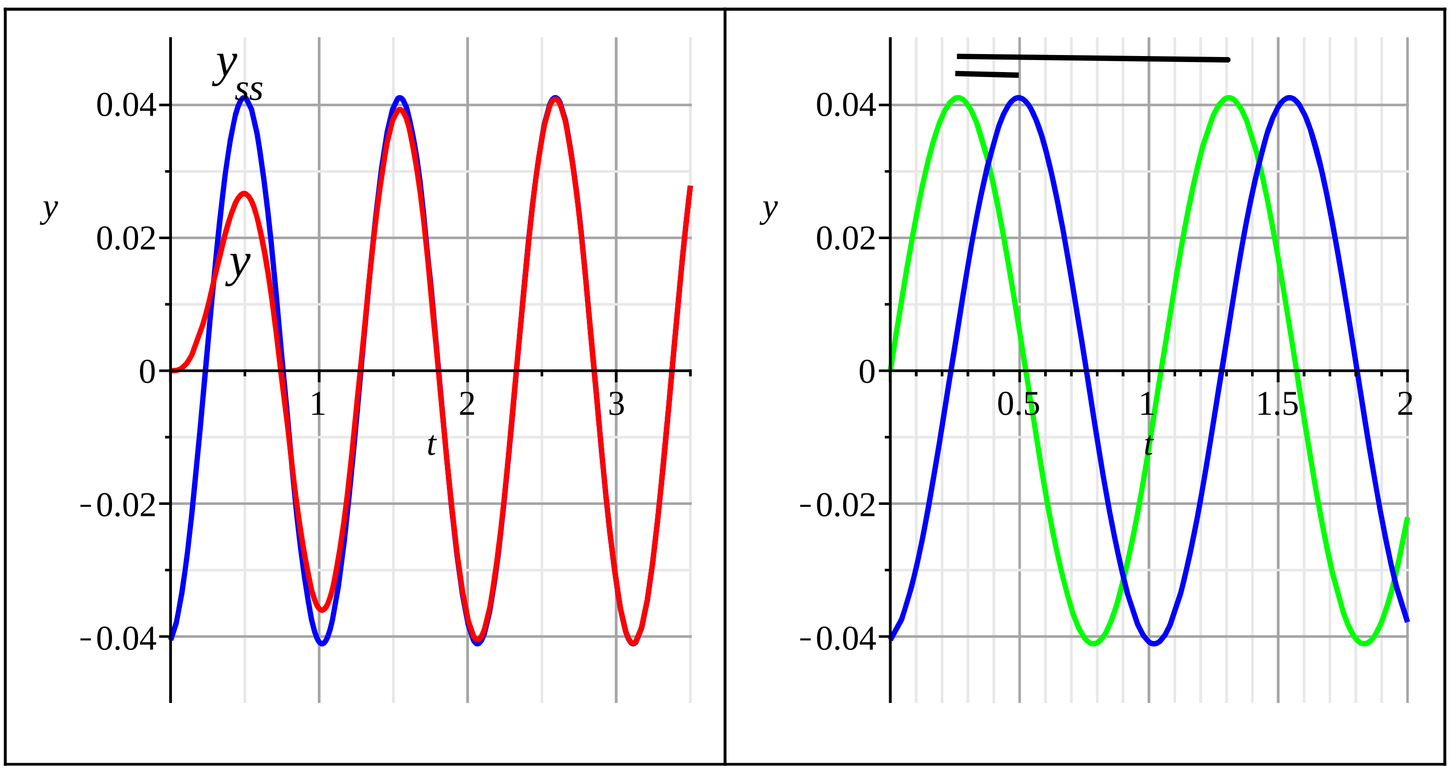
$$\boxed{y = \underbrace{-\tfrac{1}{120}}_{-1/40}(3\cos 6t + \sin 6t) + \tfrac{1}{40}}$$

MAT2705-02/06 09F Take Home Test 3 graphics

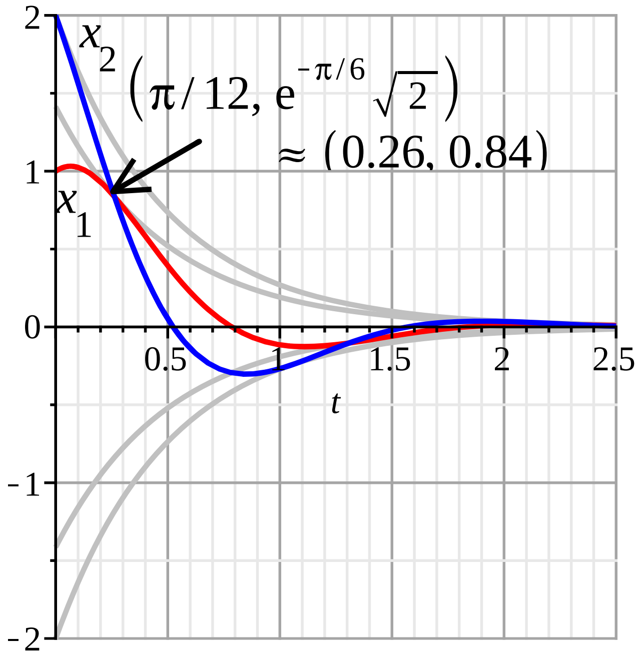
1.c), g)



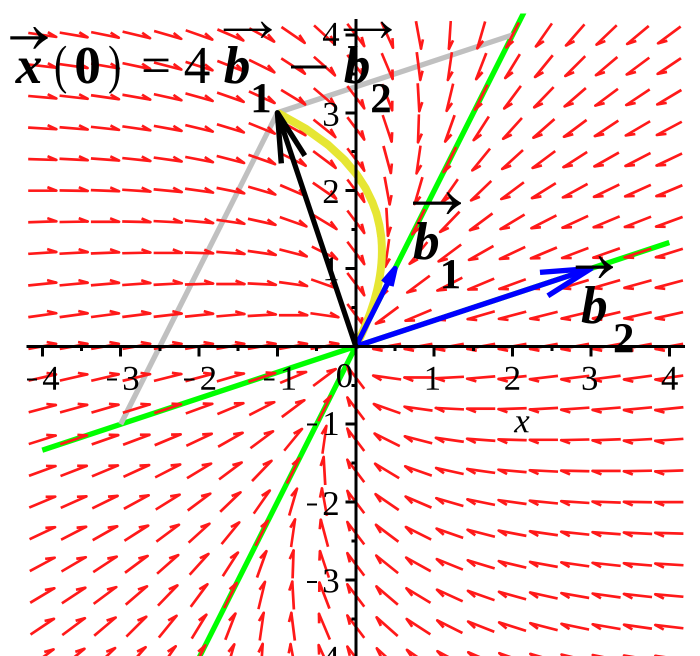

1.d)

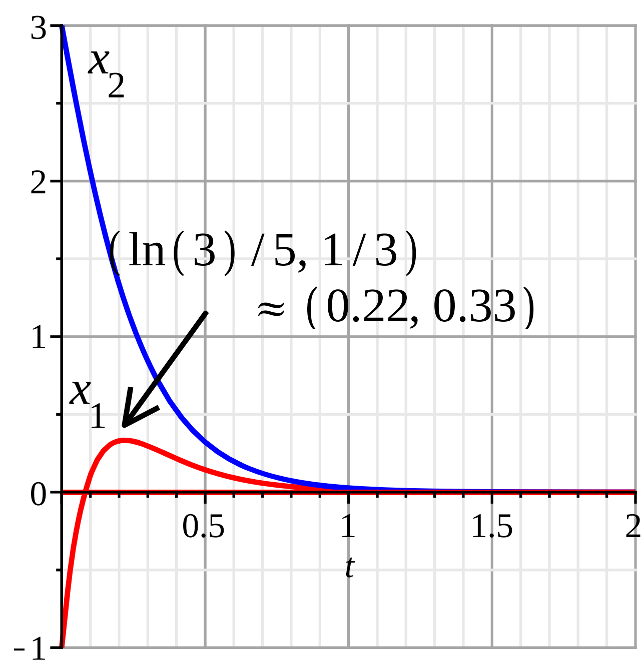


2.e) f)


$x_2$
$x_1$
$\left(\pi/12, e^{-\pi/6}\sqrt{2}\right)$
$\approx (0.26, 0.84)$
2
1
0
-1
-2
0.5
1
1.5
2
2.5
t

3.c), d)


$\vec{x}(0) = 4\,\vec{b}_1 - \vec{b}_2$
$\vec{b}_1$
$\vec{b}_2$
4
3
2
1
-1
-2
-3
-4
-4
-3
-2
-1
0
1
2
3
4
x


$x_2$
$x_1$
$(\ln(3)/5, 1/3)$
$\approx (0.22, 0.33)$
3
2
1
0
-1
0.5
1
1.5
2
t