

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses, plotting and root finding.** *Print* requested technology plots, annotate them appropriately by hand and attach to the relevant problems.

1. The displacement  $y(t)$  of an underdamped harmonic oscillator system satisfies

$$m y''(t) + c y'(t) + K y(t) = F(t).$$

Let  $m = 1$ ,  $c = 4$ ,  $K = 40$  and the initial conditions  $y(0) = 0$ ,  $y'(0) = 0$ .

a) What are the natural frequency  $\omega_0$ , natural decay time  $\tau_0$ , the quality factor  $Q = \omega_0 \tau_0$  and the period  $T_0 = 2\pi/\omega_0$  for this system? (Give both exact and numeric values to 3 decimal places.)

Consider the following driving force functions  $F(t)$ :

b)  $F(t) = 0$ .

Find the general solution of the differential equation. What is the frequency  $\omega_1$ , decay time  $\tau_1$  and the period

$T_1 = 2\pi/\omega_1$  for this decaying sinusoidal solution? (Give both exact and numeric values to 3 decimal places.)

c)  $F(t) = 1$ .

Find the initial value problem solution by hand and evaluate the asymptotic value  $y_\infty = \lim_{t \rightarrow \infty} y(t)$ .

Make a plot in an appropriate viewing window ( $t \geq 0$  up to 5 decay times) showing both the solution and its horizontal asymptote together, showing clearly the initial oscillating behavior and the approach to the asymptotic value. Does the period of the damped oscillation in your plot roughly agree with  $T_1$ ? Support your claim.

d)  $F(t) = \sin(6t)$ .

Find the initial value problem solution by hand (but check with Maple!)

Evaluate the values of the amplitude  $A$  and phase shift  $\delta$  for the steady state solution (the part of the solution which remains after the transient has died away) and express the phase shift in radians, degrees and cycles.

Make a single plot in an appropriate viewing window showing both the solution function and the steady state solution until they merge. In a separate plot for comparison with the driving sine function, plot both the steady state solution and  $A \sin(6t)$  (same amplitude as the steady state solution) to see how the peaks of the steady state solution compare to the peaks of the driving function. Does your plot agree with the phase shift you calculated in fractions of a cycle, given that the driving sine is itself shifted  $1/4$  cycle to the right (later in time) from the vertical axis at  $t = 0$ ? Explain.

e)  $F(t) = \sin(\omega t)$ .

Explore resonance for this system by finding the steady state solution by hand, where the nonnegative frequency  $\omega$  of the driving force function is a parameter.

f) Evaluate the steady state amplitude function  $A(\omega)$  and use calculus to find the exact and numerical value of the frequency  $\omega_p$  and the amplitude  $A(\omega_p)$  where it has its peak value for  $\omega \geq 0$ .

What is the numerical value of the ratio  $A(\omega_p)/A(0)$ ? How does it compare to the quality factor  $Q$ ? Does  $A(6)$  agree with  $A$  from part d)?

g) Plot this amplitude function  $A(\omega)$  in an appropriate window (showing the behavior of the entire function for  $\omega \geq 0$ ) together with the constant functions  $A(\omega_0)$ ,  $A(\omega_p)$  and  $A(0)$  and hand annotate on your axes the values of these frequencies and amplitudes and indicate the points on the curve where these values are obtained.

$$2. \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

a) Write this system of differential equations for the vector variable  $\vec{x} = \langle x_1, x_2 \rangle$  AND its initial conditions in

scalar form (use arrow notation for vectors!).

b) Use the eigenvector approach to find its general solution by hand, showing all steps.

c) Find the IVP solution, using matrix methods showing all steps. Make sure it agrees with Maple's solution.

d) Express sinusoidal factors in each of the two unknowns as phase-shifted cosines in terms of their amplitudes and phase shifts. Determine the upper and lower envelope curves for each such function from this step (plus or minus amplitude times exponential factor from solutions).

e) Make a single plot showing the original two solution curves (this is a check of your amplitude) and their envelope curves versus  $t$  on the same axes for at least 5 characteristic times of the decaying exponential factor in these expressions (i.e., in an appropriate viewing window). Hand annotate each solution curve with its variable name.

f) It appears that the first intersection point of the graphs of the two variables in this plot coincides with the point where one of the two variables touches its envelope curve for the first time. Find this common intersection point exactly and numerically to several digits accuracy and show that indeed the two functions and the relevant envelope curve all have the same exact value at that point.

Does your calculated value of  $t$  where this occurs seem consistent with your plot? Explain

$$3. x_1'(t) = -11x_1(t) + 3x_2(t), x_2'(t) = -2x_1(t) - 4x_2(t), x_1(0) = -1, x_2(0) = 3.$$

a) Identify the coefficient matrix and find a new basis for  $R^2$  consisting of eigenvectors  $\vec{b}_1, \vec{b}_2$  of this matrix using the standard hand recipe. Order the real eigenvalues  $\lambda_1 \geq \lambda_2$  by decreasing value.

b) Evaluate the new coordinates  $\langle y_1, y_2 \rangle$  of the point  $\langle x_1, x_2 \rangle = \langle -1, 3 \rangle$  with respect to this basis of eigenvectors.

c) Use technology to plot a directionfield for this DE with the solution curve through the single initial data point, and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose the window  $x_1 = -4 \dots 4, x_2 = -4 \dots 4$ . By hand label these lines by their new coordinate labels, draw in and label the eigenvectors and the initial data vector  $\vec{x}(0)$  themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., the parallelogram parallel to the new coordinate axes with the initial data vector as the main diagonal. Do the projections along the coordinate axes agree with the values you found for the new coordinates of this vector? Explain. Does your directionfield correspond to the eigenvectors you have drawn? Explain why.

d) Plot the two variables versus  $t$  for an appropriate viewing window for this initial value problem based on the longest characteristic time. Explain your window choice. The first variable  $x_1$  has a local maximum obvious in your plot. Find its coordinates and the corresponding value of  $t$  exactly and approximately. Annotate your diagram to show this point.

**Advice.** When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress me as though it were material for a job interview (you're fired! or you're hired! ?). In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

## ► solution

### ▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:  
"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature:

Date: