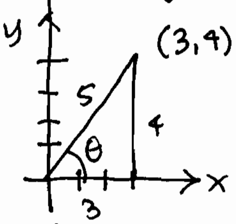
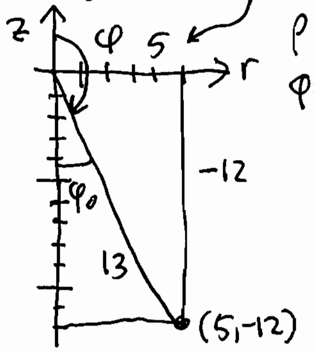
MAT2500-02 08S Final Exam Answers

① $(x,y,z) = (3,4,-12)$, $-12 = z$



$r = \sqrt{3^2+4^2} = 5$

$\theta = \arctan\frac{4}{3} \approx 53.1°$



$\rho = \sqrt{5^2+12^2} = 13$

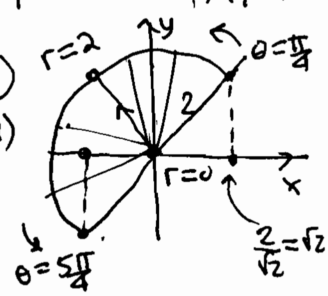
$\varphi = \pi - \arctan\frac{5}{12} \approx 157.4°$

$(\varphi_0 \approx 32.6°)$

cylindrical: $(r,\theta,z) = (5, \arctan\frac{4}{3}, -12)$

spherical: $(\rho,\varphi,\theta) = (13, \pi-\arctan(\frac{5}{12}), \arctan\frac{4}{3})$

② a)



$\boxed{\begin{array}{l} r = 0..2 \\ \theta = \frac{\pi}{4}..\frac{5\pi}{4} \end{array}}$

b) $(x,y) = (r\cos\theta, r\sin\theta)$

$A_y = \iint y\, dA = \int_{\pi/4}^{5\pi/4}\int_0^2 (r\sin\theta)\, r\, dr\, d\theta$

$= \Big(\underbrace{\int_{\pi/4}^{5\pi/4} \sin\theta\, d\theta}_{-\cos\theta\,\big|_{\pi/4}^{5\pi/4}}\Big)\Big(\underbrace{\int_0^2 r^2 dr}_{\frac{r^3}{3}\big|_0^2 = \frac{8}{3}}\Big) = \boxed{\frac{8\sqrt{2}}{3}}$

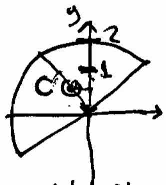
$= -(-\frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}}$

$= 2/\sqrt{2} = \sqrt{2}$

$A = \iint 1\, dA = \int_{\pi/4}^{5\pi/4}\int_0^2 1\, r\, dr\, d\theta = \boxed{2\pi}$

$= \Big(\underbrace{\int_{\pi/4}^{5\pi/4} d\theta}_{\theta\,\big|_{\pi/4}^{5\pi/4} = \pi}\Big)\Big(\underbrace{\int_0^2 r\, dr}_{\frac{r^2}{2}\big|_0^2 = 2}\Big)$

② b) $\bar{y} = \frac{A_y}{A} = \frac{8\sqrt{2}/3}{2\pi} = \frac{4\sqrt{2}}{3\pi} \approx \boxed{0.6002}$



about ½ looks right

centroid on symmetry axis of semicircle more towards base

c) On C_1 it is big & nearly tangent to the curve in the same direction, but $\vec{F}$ is small and/or perpendicular to C_2 so the contribution will be small in comparison. So the line integral will be positive.

d) C_1: $r = 2$, $\theta = \frac{\pi}{4}..\frac{5\pi}{4}$ so $x = 2\cos t$, $y = 2\sin t$ ($\theta \to t$)

$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle$ $\vec{r}\,' = \langle -2\sin t, 2\cos t\rangle$

$\vec{F}(\vec{r}(t)) = \langle -\frac{(2\sin t)^2}{2}, 2\cos t\rangle = \langle -2\sin^2 t, 2\cos t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = (-2\sin t)(-2\sin^2 t) + (2\cos t)(2\cos t)$

$= 4\sin^3 t + 4\cos^2 t$

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{\pi/4}^{5\pi/4} (4\sin^4 t + 4\cos^2 t)\, dt$

maple $\boxed{\frac{10}{3}\sqrt{2} + 2\pi}$

C_2: $y = x$ $x = -\frac{2}{\sqrt{2}}..\frac{2}{\sqrt{2}} = -\sqrt{2}..\sqrt{2}$ ($x \to t$)

$\vec{r}(t) = \langle t, t\rangle$ $t = -\sqrt{2}..\sqrt{2}$

$\vec{r}\,'(t) = \langle 1, 1\rangle$

$\vec{F}(\vec{r}(t)) = \langle -\frac{1}{2}t^2, t\rangle$

$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = -\frac{1}{2}t^2 + t$

$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{-\sqrt{2}}^{\sqrt{2}} -\frac{1}{2}t^2 + t\, dt = -\frac{1}{6}t^3 + \frac{t^2}{2}\Big|_{-\sqrt{2}}^{\sqrt{2}}$

$= -\frac{2\sqrt{2}}{6} - \frac{2\sqrt{2}}{6} + 0 = \boxed{-\frac{2}{3}\sqrt{2}}$

$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \frac{10}{3}\sqrt{2} + 2\pi - \frac{2}{3}\sqrt{2}$

$= \boxed{\frac{8\sqrt{2}}{3} + 2\pi}$

e) $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-\frac{y^2}{2}) = 1 + y$

$\int_C \vec{F}\cdot dr = \iint_R 1 + y\, dA \;\left(= \iint 1\, dA + \iint y\, dA\right)$ $\left(\int d\theta\right)\left(\int_0^2 r\,dr\right)$, $= \pi$, $= 2$

$= \int_{\pi/4}^{5\pi/4}\int_0^2 (1 + r\sin\theta)\, r\, dr\, d\theta = \Big(\underbrace{\int_{\pi/4}^{5\pi/4} \sin\theta\, d\theta}_{-\cos\theta\,\big|_{\pi/4}^{5\pi/4} = \sqrt{2} \text{ as above}}\Big)\Big(\underbrace{\int_0^2 r^2 dr}_{\frac{r^3}{3}\big|_0^2 = \frac{8}{3}}\Big)$

$= \boxed{\frac{8\sqrt{2}}{3} + 2\pi \approx 10.0544}$

should equal sum $A_y + A$ & does.

(3) a) $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+2yz & y^2 \end{vmatrix} = \Big\langle \underbrace{\underbrace{\tfrac{\partial}{\partial y}(y^2)}_{2y} - \underbrace{\tfrac{\partial}{\partial z}(x^2+2yz)}_{2y}}_{=0},\ \underbrace{\underbrace{\tfrac{\partial}{\partial z}(2xy)}_{0} - \underbrace{\tfrac{\partial}{\partial x}(y^2)}_{0}}_{0},\ \underbrace{\underbrace{\tfrac{\partial}{\partial x}(x^2+2yz)}_{2x} - \underbrace{\tfrac{\partial}{\partial y}(2xy)}_{-2x}}_{0} \Big\rangle$$

$$= \boxed{\langle 0,0,0\rangle}$$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \underbrace{\tfrac{\partial}{\partial x}(2xy)}_{2y} + \underbrace{\tfrac{\partial}{\partial y}(x^2+2yz)}_{2z} + \underbrace{\tfrac{\partial}{\partial z}(y^2)}_{0} = \boxed{2y+2z}$$

$$\operatorname{curl}\vec{G} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^{-x} & e^{-x} & 2z \end{vmatrix} = \Big\langle \underbrace{\underbrace{\tfrac{\partial}{\partial y}(2z)}_{0} - \underbrace{\tfrac{\partial}{\partial z}(ye^{-x})}_{0}}_{0},\ \underbrace{\tfrac{\partial}{\partial z}(ye^{-x})}_{0} - \underbrace{\tfrac{\partial}{\partial x}(2z)}_{0},\ \underbrace{\tfrac{\partial}{\partial x}(e^{-x}) - \tfrac{\partial}{\partial y}(ye^{-x})}_{\substack{-e^{-x}-e^{-x} \\ -2e^{-x}}} \Big\rangle$$

$$= \boxed{\langle 0, 0, -2e^{-x}\rangle}$$

$$\operatorname{div}\vec{G} = \underbrace{\tfrac{\partial}{\partial x}(ye^{-x})}_{-ye^{-x}} + \underbrace{\tfrac{\partial}{\partial y}(e^{-x})}_{0} + \underbrace{\tfrac{\partial}{\partial z}(2z)}_{2} = \boxed{2-ye^{-x}}$$

b) $\vec{F}$ is conservative since $\operatorname{curl}\vec{F} = \vec{0}$.

(4) a) $f = 3x^2+2xy+3y^2$

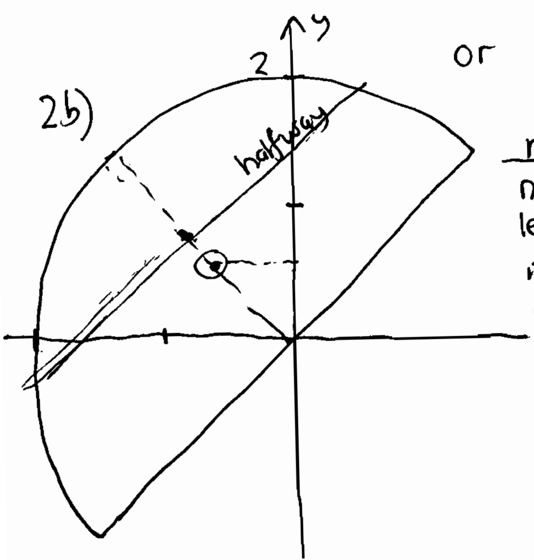
$$\vec{\nabla}f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \cancel{\frac{\partial f}{\partial z}} \right\rangle = \Big\langle \underbrace{\tfrac{\partial}{\partial x}(3x^2+2xy+3y^2)}_{6x+2y+0},\ \underbrace{\tfrac{\partial}{\partial y}(3x^2+2xy+3y^2)}_{0+2x+6y} \Big\rangle$$

$$= \boxed{\langle 6x+2y,\ 2x+6y\rangle}$$

b) $$\int_C \vec{F}\cdot d\vec{r} = f(\tfrac{1}{2},\tfrac{1}{2}) - f(-\tfrac{1}{2},\tfrac{1}{2}) = 3(\tfrac{1}{2})^2 + 2(\tfrac{1}{2})(\tfrac{1}{2}) + 3(\tfrac{1}{2})(\tfrac{1}{2}) - [3(\tfrac{1}{2})^2 + 2(-\tfrac{1}{2})(\tfrac{1}{2}) + 3(\tfrac{1}{2})(\tfrac{1}{2})] = \tfrac{1}{2}+\tfrac{1}{2} = 1$$

$$\text{or } = \frac{3+2+3}{4} - \left[\frac{3-2+3}{4}\right] = 2-1 = \boxed{1}$$

2b)



revisited:
now that I have extra space here let me redraw my centroid guess. it should be below the halfway point on the semicircle symmetry axis which looks exactly like 0.6 on the axis, eh?