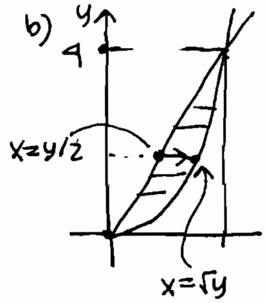
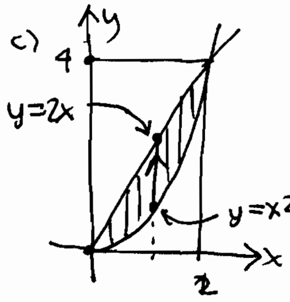
MAT2500-02 08S TakeHome Test 3 Answers

(1) a) $\displaystyle \underbrace{\int_{y=0}^{y=4}\int_{x=y/2}^{x=\sqrt{y}}}\ e^{3x^2-x^3}\,dx\,dy \approx 17.866050$ (maple)

b) 

$y=0..4$

c) 

$x=0..2$

d) $\displaystyle \underbrace{\int_0^2\int_{x^2}^{2x}}\ e^{3x^2-x^3}\,dy\,dx$

e) $\displaystyle \int_0^2 e^{3x^2-x^3}\Big(\underbrace{\int_{x^2}^{2x} 1\,dy}_{y|_{x^2}^{2x}=2x-x^2}\Big)dx$

$\displaystyle =\int_0^2 e^{\overbrace{3x^2-x^3}^{u}}\underbrace{(2x-x^3)\,dx}_{du/3}$

$u=3x^2-x^3$

$du=(6x-3x^2)dx=3(2x-x^2)dx$

$\displaystyle =\int_{x=0}^{x=2}\frac{1}{3}e^u\,du=\frac{1}{3}e^u\Big|_{x=0}^{x=2}$

$\displaystyle =\frac{1}{3}e^{3x^2-x^3}\Big|_0^2=\frac{1}{3}\left[e^{12-8}-e^0\right]$

$=\boxed{\frac{1}{3}(e^4-1)} \approx 17.866050$ ✓

(2) $\displaystyle \underbrace{\int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{2x-x^2}}}\ \underbrace{\sqrt{x^2+y^2}}_{r}\,dy\,dx$

$y^2=2x-x^2$

$x^2+y^2=2x$

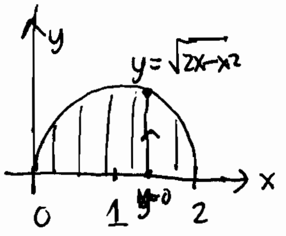
$x^2-2x+y^2=0$

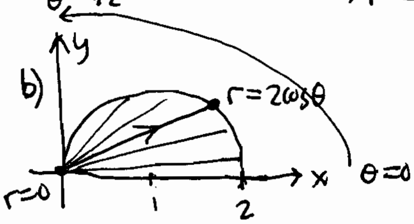
$(x-1)^2+y^2=1$

Circle at (1,0) radius 1

$r^2=2(r\cos\theta)$

$r=2\cos\theta$

a) 

b) 

(2) c) $\displaystyle \int_0^{\pi/2}\int_0^{2\cos\theta}\underbrace{(r)}_{\sqrt{x^2+y^2}}\underbrace{(r\,dr\,d\theta)}_{dA=dy\,dx}$

d) $\displaystyle =\int_0^{\pi/2}\underbrace{\int_0^{2\cos\theta} r^2\,dr}_{\frac{r^3}{3}\Big|_{r=0}^{r=2\cos\theta}=\frac{8\cos^3\theta}{3}}\,d\theta=\frac{8}{3}\int_0^{\pi/2}\cos^3\theta\,d\theta$

$\displaystyle =\frac{8}{3}\left[\frac{1}{3}\cos^2\theta\sin\theta+\frac{2}{3}\sin\theta\right]_0^{\pi/2}=\frac{8}{3}\left[0+\frac{2}{3}(1)-0-0\right]=\boxed{\frac{16}{9}}$

$\approx 1.777778 \approx$ numerical value of original integral. ✓
yes, they agree!

(3) a) $\displaystyle \int_0^2\int_0^{y^3}\underbrace{\int_0^{y^2} z\,dz}_{\frac{1}{2}z^2|_0^{y^2}=\frac{1}{2}y^4-0=\frac{y^4}{2}}dx\,dy=\int_0^2\underbrace{\int_0^{y^3}\frac{y^4}{2}dx}_{\frac{1}{2}y^4x|_{x=0}^{x=y^3}=y^7}dy=\int_0^2\frac{y^7}{2}dy$

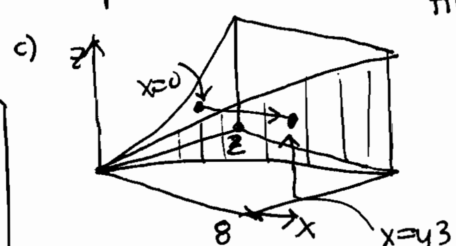
$\displaystyle =\frac{y^8}{16}\Big|_0^2=\frac{2^8}{2^4}-0=2^4=\boxed{16}\equiv V_z$

$\displaystyle \int_0^2\int_0^{y^3}\underbrace{\int_0^{y^2}1\,dz}_{z|_0^{y^2}=y^2}dx\,dy=\int_0^2 y^5\,dy=\frac{y^6}{6}\Big|_0^2=\frac{2^6}{6}=\frac{2^5}{3}=\boxed{\frac{32}{3}}\equiv V$

(with $\displaystyle y^2x\Big|_{x=0}^{x=y^3}=y^5$)

$\displaystyle \bar z=\frac{V_z}{V}=\frac{16}{32/3}=\boxed{\frac{3}{2}}$

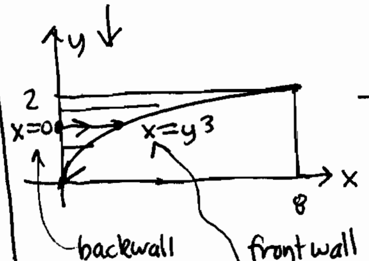
b) Maple agrees!

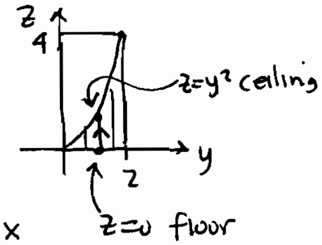
c) 

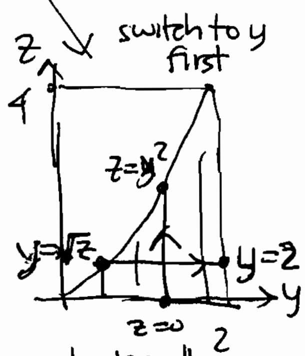
first $x=0..?$
backwall
frontwall see below $x=y^3$

then

$\displaystyle \underbrace{\int_{y=0}^{y=2}\int_{x=0}^{x=y^3}\int_{z=0}^{z=y^2}}\ f\,dz\,dx\,dy$







backwall projection
while $z=0..4$.

so $\displaystyle \boxed{\int_0^4\int_{\sqrt z}^2\int_0^{y^3} f\,dx\,dy\,dz}$ for $f=z,1$

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③ d) $V = \int_0^4 \int_{\sqrt{z}}^2 \int_0^{y^3} 1\, dx\,dy\,dz = \int_0^4 \int_{\sqrt{z}}^2 y^3\,dy\,dz = \int_0^4 \frac{16-z^2}{4}dz = \frac{16z - z^3/3}{4}\Big|_0^4 = \frac{64-64/3}{4}$

$\underbrace{\int_0^{y^3} 1\,dx}_{x\big|_{x=0}^{x=y^3} = y^3}$ $\qquad \underbrace{\int_{\sqrt z}^2 y^3\,dy}_{\frac{y^4}{4}\big|_{y=\sqrt z}^{y=2} = \frac{16-z^2}{4}}$ $\qquad = 16\left(\frac{2}{3}\right) = \boxed{\frac{32}{3}}$ ✓ yes!

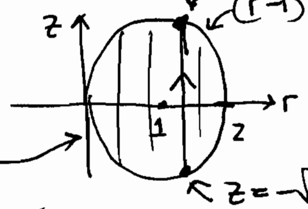
$V_z = \int_0^4 \int_{\sqrt z}^2 \int_0^{y^3} z\,dx\,dy\,dz = \int_0^4 \frac{z}{4}(2^4 - z^2)\,dz = \int_0^4 \frac{1}{4}(2^4 z - z^3)\,dz = \frac{1}{4}\left(2^4\frac{z^2}{2} - \frac{z^4}{4}\right)\Big|_0^4$

$\underbrace{\int_0^{y^3} z\,dx}_{zy^3}$, $\quad zy^4/4\Big|_{y=\sqrt z}^{y=2}$

$= \frac{1}{4}(2^3 \cdot 16 - 4^3) = \frac{64}{4} = \boxed{16}$ ✓ yes!

$2\cdot 64 - 64$

④ a)



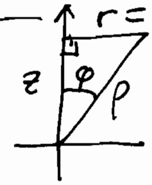
$z = \sqrt{2r - r^2}$

$(r-1)^2 + z^2 = 1 \rightarrow r^2 - 2r + 1 + z^2 = 1$

$\underbrace{r^2 + z^2}\ -2r + \cancel{1} = \cancel{1}$

$\rho^2 = 2r = 2\rho\sin\varphi$

$\boxed{\rho = 2\sin\varphi}$



$r = \rho\sin\varphi$

$z = -\sqrt{2r - r^2}$

b) $z^2 = 2r - r^2$

$z = \pm\sqrt{2r - r^2}$

b) $r = 0..2$

$\text{Int} = \int_0^{2\pi}\int_0^2 \underbrace{\int_{-\sqrt{2r-r^2}}^{\sqrt{2r-r^2}} z^2\, r\, dz\,dr\,d\theta}$ $= \int_0^{2\pi}\int_0^2 \frac{2r}{3}(2r-r^2)^{3/2}\,dr\,d\theta$

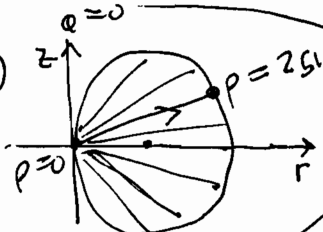
$\hat{r}$ correction factor

$\frac{rz^3}{3}\Big|_{z=-\sqrt{2r-r^2}}^{z=\sqrt{2r-r^2}} = \frac{2r}{3}(2r-r^2)^{3/2}$

$= \int_0^{2\pi} -\frac{2}{15}(2r-r^2)^{5/2} - \frac{1}{12}\underbrace{(2-2r)(2r-r^2)^{3/2}}_{=(2r-r^2)^{5/2}} - \frac{1}{8}(2-2r)(2r-r^2)^{1/2} + \frac{1}{4}\arcsin(r-1)\Big|_{r=0}^{r=2}\, d\theta$

$= 2\pi\left[\cancel{0+\ldots}\quad 0 + \frac{1}{4}[\arcsin(1) - \arcsin(-1)]\right] = \frac{\pi}{2}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \boxed{\frac{\pi^2}{2}}$

b)



$\varphi = 0$

$\rho = 2\sin\varphi$

$\rho = 0$

$\varphi = 0..\pi$

d) $V = \int_0^{2\pi}\int_0^{\pi} \underbrace{\int_0^{2\sin\varphi} 1\, \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta}$ $= (2\pi)\int_0^{\pi}\frac{8}{3}\sin^4\varphi\,d\varphi$

$\rho^2\sin\varphi$ correction factor

$\frac{\rho^3}{3}\sin\varphi\Big|_{\rho=0}^{\rho=2\sin\varphi} = \frac{8}{3}\sin^4\varphi$

$= \frac{16\pi}{3}\int_0^{\pi}\sin^4\varphi\,d\varphi = \frac{16\pi}{3}\left[-\frac{1}{4}\sin^3\varphi\cos\varphi - \frac{3}{8}\cos\varphi\sin\varphi + \frac{3}{8}\varphi\right]_0^{\pi}$

$= \frac{16\pi}{3}\left[0 + \frac{3}{8}(\pi)\right] = \boxed{2\pi^2}$

agree!

$V_{z^2} = \int_0^{2\pi}\int_0^{\pi}\int_0^{2\sin\varphi} \overbrace{(\rho\cos\varphi)^2}^{z^2}\, \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\cdot\frac{2^5}{5}\int_0^{\pi}\sin^6\varphi\cos^2\varphi\,d\varphi = \cancel{\frac{64\pi}{5}}\left(\cancel{\frac{5\pi}{128}}\right) = \boxed{\frac{\pi^2}{2}}$ ✓

$\underbrace{\rho^4\cos^2\varphi\sin\varphi}$

$\frac{\rho^5}{5}\Big|_{\rho=0}^{\rho=2\sin\varphi}\cos^2\varphi\sin\varphi = \frac{2^5}{5}\sin^6\varphi\cos^2\varphi$

$-\frac{1}{8}\sin^5\varphi\cos^3\varphi - \frac{5}{48}\sin^3\varphi\cos^3\varphi - \frac{5}{64}\sin\varphi\cos^3\varphi + \frac{5}{128}\cos\varphi\sin\varphi + \frac{5}{128}\varphi\Big|_0^{\pi} = 0 + \frac{5}{128}\pi$

e) $\overline{z^2} = \frac{V_{z^2}}{V} = \frac{\pi^2/2}{2\pi^2} = \boxed{\frac{1}{4}}$ ✓

okay, it's a bit sloppy! but legible I hope.