MAT2705-01 08F Quiz 8 Print Name (Last, First) ________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC). **You may use technology to find the roots of polynomials. [No mercy for mistakes here.]**

1. $y'' + 4\,y' + 8\,y = \cos(2\,t)\ ,\ y(0) = \frac{1}{2}\ ,\ y'(0) = 0$

a) Find the general solution y_h of the related homogeneous DE. Is this system underdamped, critically damped or underdamped and why?

b) Use the method of undetermined coefficients to find the steady state sinusoidal solution (the particular solution y_p) of this driven harmonic oscillator DE.

c) Find the solution of the initial value problem.

d) Evaluate the amplitude A and phase shift δ of the steady state solution function, and numerically evaluate these formulas, and evaluate the fraction $\delta/(2\pi)$ to see what fraction of a cycle the phase is shifted. What is the period T of the steady state solution?

e) Identify the values of the natural decay constant k_0, the corresponding characteristic time τ_0, the natural frequency ω_0, and the quality factor $Q = \omega_0 \tau_0$.

f) Show that the amplitude agrees with the formula

(2) oops

$$A(\omega) = \left(\left(\omega^2 - \omega_0\right)^2 + k_0^2\,\omega^2 \right)^{-\frac{1}{2}} \quad \text{with } \omega = 2.$$

g) **Optional.** Use technology to plot both the solution and the steady state solution together on the same axes as well as $A, -A$ for an appropriate window (say 2 periods of the steady state solution) in which one sees clearly the merging of the two curves as the solution reaches the steady state at least to the pixel size of your technology grapher. Sketch what you see, labeling the axes, tickmarks, etc. Does your steady state solution agree with your amplitude envelope? Does the phase shift fraction of a full cycle look right? Does the time interval it takes for the difference between the two solutions to disappear agree with the characteristic time of the homogeneous solution (called the transient). Explain. → see maple worksheet for part g) to see how to visualize these formulas

► **solution**

a) $y = e^{rt} \to \text{DE} \to r^2 + 4r + 8 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4\cdot 8}}{2} = -2 \pm 2i$

$e^{rt} = e^{(-2\pm 2i)t} = e^{-2t}(\cos 2t \pm i \sin 2t)$

$\hookrightarrow e^{-2t}\cos 2t,\ e^{-2t}\sin 2t$

$$\boxed{y_h = \underbrace{e^{-2t}(c_1 \cos 2t + c_2 \sin 2t)}_{}}$$

damped oscillation means "underdamped"

b) $\cos 2t \to r = \pm 2i \to e^{rt} = e^{\pm 2it} \hookrightarrow \cos 2t, \sin 2t$

no root interference between LHS, RHS so trial function is a general solution of $(D^2+4)y_p = 0$ corresponding to $r^2 + 4 = 0$:

$8\,[y_p = c_3 \cos 2t + c_4 \sin 2t]$

$4\,[y_p' = -2c_3 \sin 2t + 2c_4 \cos 2t]$

$1\,[y_p'' = -4c_3 \cos 2t - 4 c_4 \sin 2t]$

$y_p'' + 4y_p' + 8y_p = [(8-4)c_3 + 8c_4]\cos 2t = 1 \cos 2t$
$\quad + [-8c_3 + (8-4)c_4]\sin 2t \quad + 0 \sin 2t$

b) continued:

$$\begin{bmatrix} 4 & 8 \\ -8 & 4 \end{bmatrix}\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{16+64}\begin{bmatrix} 4 & -8 \\ 8 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{80}\begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 1/20 \\ 1/10 \end{bmatrix}$$

$\boxed{y_p = \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t}$ steady state solution

c) $y = y_h + y_p = e^{-2t}(c_1 \cos 2t + c_2 \sin 2t) + \frac{1}{20}\cos 2t + \frac{1}{10}\sin 2t$

$y' = -2e^{-2t}(c_1 \cos 2t + c_2 \sin 2t) - \frac{2}{20}\sin 2t + \frac{2}{10}\cos 2t$
$\quad + e^{-2t}(-2c_1 \sin 2t + 2c_2 \cos 2t)$

$y(0) = c_1 + \frac{1}{20} = 1/2 \to c_1 = \frac{10}{20} - \frac{1}{20} = \frac{9}{20}$

$y'(0) = -2c_1 + 2c_2 + \frac{2}{10} = 0 \to c_2 = -\frac{1}{10} + c_1 = -\frac{2}{20} + \frac{9}{20} = \frac{7}{20}$

$$\boxed{y = \frac{1}{20}e^{-2t}(9\cos 2t + 7\sin 2t) + \frac{1}{20}(\cos 2t + 2\sin 2t)}$$

d) $(c_3, c_4) = \frac{1}{20}(1, 2) \quad A = \frac{1}{20}\sqrt{1+4} = \frac{\sqrt{5}}{20} \approx 0.112$

same!

$\tan\delta = \frac{2}{1} \to \delta = \arctan 2 \approx 1.11 \approx 63^\circ \quad \frac{\delta}{2\pi} = \frac{(\arctan 2)}{2\pi} \approx 0.176$

$\boxed{T = \frac{2\pi}{2} = \pi}$

e) $y'' + k_0 y + \omega_0^2 y = \text{LHS} \to k_0 = 4,\ \tau_0 = 1/4 = 0.25$

$\omega_0 = \sqrt{8} = 2\sqrt{2} \approx 2.83 \qquad Q = \frac{1}{4}\sqrt{8} = \frac{\sqrt{2}}{2} \approx 0.707$

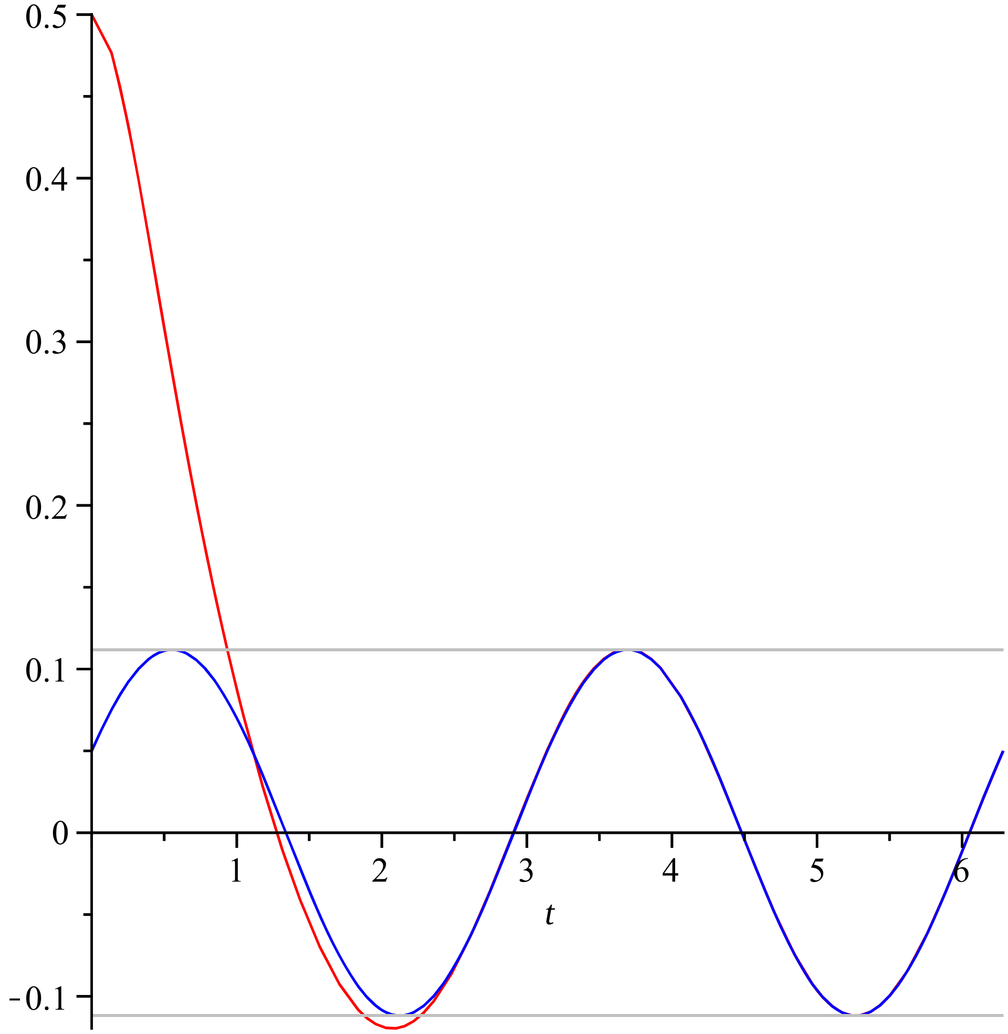
f) $A(2) = [(2^2 - 8)^2 + 4^2 \cdot 2^2]^{-1/2} = [16 + 16\cdot 4]^{-1/2} = \frac{1}{4\sqrt{5}} = \frac{\sqrt{5}}{20}$

↑ sorry about misprint, I will disregard value here as long as you properly substituted k_0, ω_0 ($\to A \approx 0.124$)

> $A, evalf(A)$

$$\frac{1}{40}\sqrt{5}\sqrt{4}, 0.1118033988 \tag{1.14}$$

> $plot\left(\left[\frac{7}{20}e^{-2t}\sin(2t)+\frac{9}{20}e^{-2t}\cos(2t)+\frac{1}{10}\sin(2t)+\frac{1}{20}\cos(2t), \frac{1}{10}\sin(2t)+\frac{1}{20}\cos(2t), A, -A\right], t=0..2\pi, color=[red, blue, gray, gray]\right)$



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The calculated amplitude (gray horizontal lines) indeed describes the steady state solution (blue).

The characteristic decay time for the transient is 1/2 and clearly by 5 such characteristic times, the transient (difference between red and blue curves) has disappeared in the graphic as the full solution curve (red) merges with the steady state solution (blue).

The positive phase shift by about 0.17 cycles (roughly 1/6 of a cycle) shifts the graph to the right by about that amount.