MAT 2705-01 08F Takehome Test 3 Answers (1)

① a) $$y''+\underbrace{2}_{k_0}y'+\underbrace{26}_{\omega_0^2}y=F$$

$k_0=2\to\tau_0=k_0^{-1}=1/2=0.5$

$\omega_0=\sqrt{26}\approx5.099$ $\quad T_0=\frac{2\pi}{\omega_0}=\frac{2\pi}{\sqrt{26}}\approx1.232$

$Q=\frac{\sqrt{26}}{2}\approx2.550$

b) $y\sim e^{rt}\to y''+2y'+26y=0\to$

$r^2+2r+26=0$

$r=\frac{-2\pm\sqrt{4-4(26)}}{2}=-1\pm5i$

$e^{rt}=e^{-t}e^{\pm5it}=e^{-t}(\cos5t\pm i\sin5t)$

$\underset{\text{real basis}}{\hookrightarrow}\quad e^{-t}\cos5t,\ e^{-t}\sin5t$

$\boxed{y=e^{-t}(c_1\cos5t+c_2\sin5t)}=y_h$

c) $F=1\to y_p=c_3$ (gen soln of $Dy_p=0$ since $DF=0$)

$\underbrace{c_3''}_{0}+2\underbrace{c_3'}_{0}+26c_3=1\to c_3=\frac{1}{26}=y_p$

$y=y_h+y_p=e^{-t}(c_1\cos5t+c_2\sin5t)+\frac{1}{26}$

$y'=-e^{-t}(c_1\cos5t+c_2\sin5t)+e^{-t}(-5c_1\sin5t+5c_2\cos5t)$

$y(0)=c_1+\frac{1}{26}=0\to c_1=-1/26$

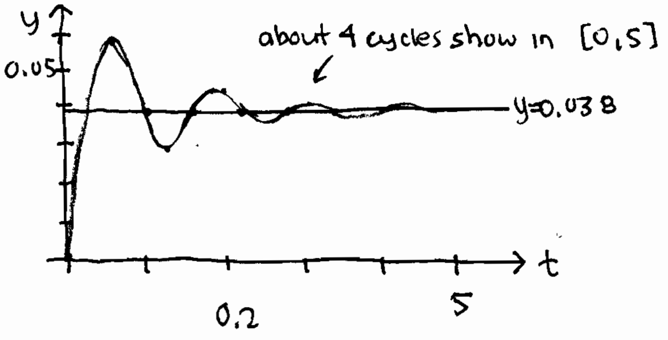
$y'(0)=-c_1+5c_2=0\to c_2=\frac{1}{5}c_1=-\frac{1}{130}$

$$\boxed{y=e^{-t}\left(-\frac{1}{26}\cos5t-\frac{1}{130}\sin5t\right)+\frac{1}{26}}$$

$y_\infty=\lim_{t\to\infty}y=\boxed{\frac{1}{26}\approx0.0384}$

$e^{-t}\to\tau=1,\ 5\tau=5$ plot $t=0..5$

$T=\frac{2\pi}{5}\approx1.257$ # oscillations: $\frac{5}{1.257}\approx3.97$



about 4 cycles show in [0,5]

d) $F=\cos(5t)\to(D^2+25)F=0\to$

$26\,[\,y_p=c_3\cos5t+c_4\sin5t\,]$

$2\,[\,y_p'=-5c_3\sin5t+5c_4\cos5t\,]$

$1\,[\,y_p''=-25c_3\cos5t-25c_4\sin5t\,]$

② d) continued

$$[\underbrace{(26-25)c_3+10c_4}_{=1}]\cos5t+[\underbrace{-10c_3+(26-25)c_4}_{=0}]\sin5t=\cos5t$$

$c_3+10c_4=1$

$-10c_3+c_4=0$

$\begin{bmatrix}1&10\\-10&1\end{bmatrix}\begin{bmatrix}c_3\\c_4\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\quad\begin{bmatrix}c_3\\c_4\end{bmatrix}=\frac{1}{101}\begin{bmatrix}1&-10\\10&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1/101\\10/101\end{bmatrix}$

$y_p=\frac{1}{101}[\cos5t+10\sin5t]$

$y=y_h+y_p=e^{-t}(c_1\cos5t+c_2\sin5t)+\frac{1}{101}(\cos5t+10\sin5t)$

$y'=-e^{-t}(c_1\cos5t+c_2\sin5t)+e^{-t}(-5c_1\sin5t+5c_2\cos5t)+\frac{1}{101}(-5\sin5t+50\cos5t)$

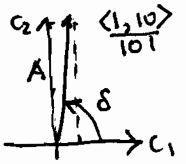
$y(0)=c_1+\frac{1}{101}=0\to c_1=-\frac{1}{101}$

$y'(0)=-c_1+5c_2+\frac{50}{101}=0\to c_2=\frac{1}{5}(c_1-\frac{50}{101})=-\frac{51}{505}$

$$\boxed{\begin{aligned}y=&\,e^{-t}\left(-\frac{1}{101}\cos5t-\frac{51}{101}\sin5t\right)\\&+\frac{1}{101}(\cos5t+10\sin5t)\end{aligned}}$$

transient

y_{ss}

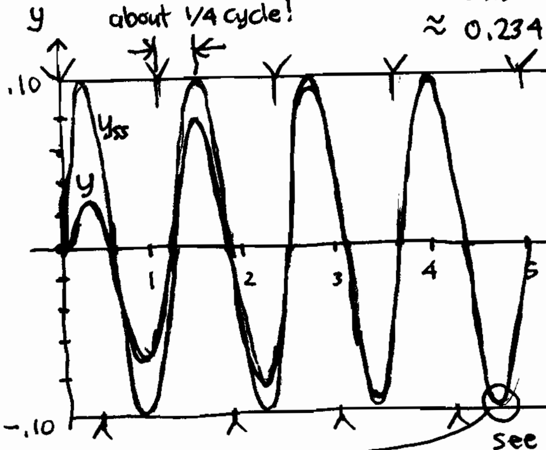


$A=\frac{1}{101}\sqrt{1^2+10^2}=\frac{\sqrt{101}}{101}=\frac{1}{\sqrt{101}}\approx0.100$

$\tan\delta=\frac{10}{1}\quad\delta=\arctan10\approx1.471$ (radians)

$\approx84.3^\circ$ ($\times\frac{180^\circ}{\pi}$)

≈0.234 cycles ($/360^\circ$)



peaks and valleys of $A\cos5t$ shown by ⋎ ⋏

to avoid confusing handsketch

see maple worksheet

slight difference detectible in last peak but transient (difference) is not detectible on screen after this

e) ↓

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① e) $F = \cos\omega t$, $(D^2+\omega^2) F = 0 \to (D^2+\omega^2) y_p = 0$

$$26\ [y_p = c_3\cos\omega t + c_4\sin\omega t\ ]$$
$$2\ [y_p' = -c_3\omega\sin\omega t + c_4\omega\cos\omega t]$$
$$\underline{1\ [y_p'' = -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t]}$$

$$y_p''+2y_p'+26y_p = [(26-\omega^2)c_3 + 2\omega c_4]\overset{\to 1}{\cos\omega t} + [-2\omega c_3 + (26-\omega^2)c_4]\underset{\to 0}{\sin\omega t} = \cos\omega t \longrightarrow$$

$$\begin{bmatrix} (26-\omega^2) & 2\omega \\ -2\omega & (26-\omega^2)\end{bmatrix}\begin{bmatrix} c_3 \\ c_4\end{bmatrix} = \begin{bmatrix} 1 \\ 0\end{bmatrix}$$

$$\begin{bmatrix} c_3 \\ c_4\end{bmatrix} = \frac{1}{(26-\omega^2)^2+4\omega^2}\begin{bmatrix} 26-\omega^2 & -2\omega \\ 2\omega & 26-\omega^2\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix}$$

$$= \frac{1}{(26-\omega^2)^2+4\omega^2}\begin{bmatrix} 26-\omega^2 \\ 2\omega\end{bmatrix}$$

$$\boxed{y_p = \frac{1}{(26-\omega^2)^2+4\omega^2}\left[(26-\omega^2)\cos\omega t + 2\omega\sin\omega t\right]}$$

f) $$A(\omega) = \frac{1}{(26-\omega^2)^2+4\omega^2}\cdot\sqrt{(26-\omega^2)^2+4\omega^2}$$

$$= \boxed{\frac{1}{\sqrt{(26-\omega^2)^2+4\omega^2}} = \frac{1}{\sqrt{676-48\omega^2+\omega^4}}}$$

$$0 = A'(\omega) = \left[\left((26-\omega^2)^2+4\omega^2\right)^{-1/2}\right]'$$
$$= -\frac{1}{2}[\cdots]^{-3/2}\left[\underbrace{2(26-\omega^2)^1(0-2\omega)+8\omega}_{4\omega\,(\underbrace{-26+\omega^2+2}_{\omega^2-24})}\right] = 0 \to$$

$$\omega_p = \sqrt{24} = \sqrt{4\cdot 6} = \boxed{2\sqrt{6} \approx \boxed{4.899}}$$

$$A(\sqrt{24}) = \frac{1}{\sqrt{(26-24)^2+4(24)}} = \frac{1}{(4+4(24))^{1/2}} = \frac{1}{2.5} = \boxed{\frac{1}{10}}$$

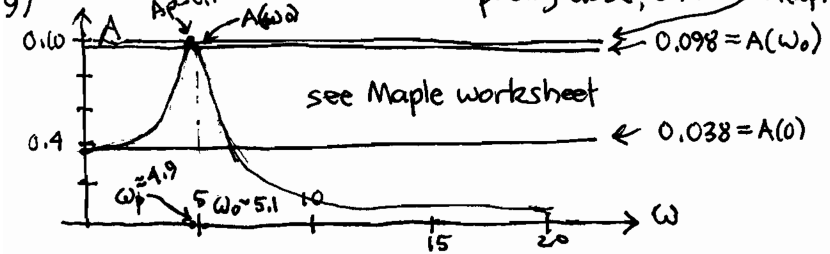
$$A(0) = \frac{1}{\sqrt{26^2}} = \frac{1}{26} = y_\infty \text{ (part c)} \quad ✓$$

$$A(5) = \frac{1}{\sqrt{(26-25)^2+4\cdot 25}} = \frac{1}{\sqrt{101}} \quad ✓ \text{ part d)}$$

$$\frac{A(\omega_p)}{A(0)} = \frac{1/10}{1/26} = \frac{26}{10} = \frac{13}{5} = 2.6 \leftrightarrow Q = 2.55$$

pretty close, eh?

g)



② a) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix}' = \underbrace{\begin{bmatrix} -3 & 0 & 1 \\ 3 & -2 & 0 \\ 0 & 2 & -1\end{bmatrix}}_{A}\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0)\end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 0\end{bmatrix}$$

b) $$|A-\lambda I| = \begin{vmatrix} -3-\lambda & 0 & 1 \\ 3 & -2-\lambda & 0 \\ 0 & 2 & -1-\lambda\end{vmatrix} \overset{\text{Maple}\downarrow}{=} -\lambda^3-6\lambda^2-11\lambda = -\lambda(\lambda^2+6\lambda+11) = 0$$

$\lambda = 0, -3\pm\sqrt{2}\,i$ (Maple or quad. formula)

$\lambda = 0$:
$$A = \begin{bmatrix} -3 & 0 & 1 \\ 3 & -2 & 0 \\ 0 & 2 & -1\end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} L & L & F \end{array} \\ \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0\end{bmatrix}\end{array}\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix} \begin{array}{l} \to x_1 = \frac{1}{3}t \\ \to x_2 = \frac{1}{2}t \end{array}$$
$\hookrightarrow x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix} = t\underbrace{\begin{bmatrix} 1/3 \\ 1/2 \\ 1\end{bmatrix}}_{\vec{b}_1}$$

$$\lambda = -3+\sqrt{2}i:\ |A-\lambda I| = \begin{vmatrix} -3-(-3+\sqrt{2}i) & 0 & 1 \\ 3 & -2-(-3+\sqrt{2}i) & 0 \\ 0 & 2 & -1-(-3+\sqrt{2}i)\end{vmatrix}$$

$$= \begin{bmatrix} -\sqrt{2}i & 0 & 1 \\ 3 & 1-\sqrt{2}i & 0 \\ 0 & 2 & 2-\sqrt{2}i\end{bmatrix} \overset{\text{Maple}\downarrow}{\Longrightarrow} \begin{array}{c} \begin{array}{ccc} L & L & F \end{array} \\ \begin{bmatrix} 1 & 0 & \frac{\sqrt{2}i}{2} \\ 0 & 1 & \frac{2-i\sqrt{2}}{2} \\ 0 & 0 & 0\end{bmatrix}\end{array}\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix}$$

$x_3 = t \quad x_1 = -\frac{\sqrt{2}i}{2}t \quad x_2 = -\frac{(2-i\sqrt{2})}{2}t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3\end{bmatrix} = t\underbrace{\begin{bmatrix} -\sqrt{2}i/2 \\ (-2+i\sqrt{2})/2 \\ 1\end{bmatrix}}_{\vec{b}_2} \qquad \vec{b}_3 = \overline{\vec{b}_2} = \begin{bmatrix} \sqrt{2}i/2 \\ (-2-i\sqrt{2})/2 \\ 1\end{bmatrix}$$

$$B = \begin{bmatrix} 1/3 & -\sqrt{2}i/2 & \sqrt{2}i/2 \\ 1/2 & (-2+i\sqrt{2})/2 & (-2-i\sqrt{2})/2 \\ 1 & 1 & 1\end{bmatrix}$$

$$\vec{x} = B\vec{y},\ \vec{y} = B^{-1}\vec{x}:\ A_D = B^{-1}AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3+\sqrt{2}i & 0 \\ 0 & 0 & -3-\sqrt{2}i\end{bmatrix}$$

$\vec{y}\,' = A_D\vec{y}$:

$$y_1' = 0 \qquad y_1 = c_1$$
$$y_2' = (-3+\sqrt{2}i)y_2 \qquad y_2 = c_2 e^{(-3+\sqrt{2}i)t}$$
$$y_3' = (-3-\sqrt{2}i)y_3 \qquad y_3 = c_3 e^{(-3-\sqrt{2}i)t}$$

$$\vec{x} = c_1\vec{b}_1 + c_2\underbrace{e^{(-3+\sqrt{2}i)t}\vec{b}_2} + \text{c.c.}$$

$$= e^{-3t}(\cos\sqrt{2}t + i\sin\sqrt{2}t)\begin{bmatrix} -\sqrt{2}i/2 \\ (-2+i\sqrt{2})/2 \\ 1\end{bmatrix}$$

$$= e^{-3t}\begin{bmatrix} +\frac{\sqrt{2}}{2}\sin\sqrt{2}t - \frac{\sqrt{2}}{2}i\cos\sqrt{2}t \\ \frac{-2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t}{2} + \frac{-2\sin\sqrt{2}t + i\sqrt{2}\cos\sqrt{2}t}{2} \\ \cos\sqrt{2}t + i\sin\sqrt{2}t\end{bmatrix}$$

$$= e^{-3t}\begin{bmatrix} \frac{\sqrt{2}}{2}\sin\sqrt{2}t \\ -\cos\sqrt{2}t - \frac{\sqrt{2}}{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t\end{bmatrix} + i e^{-3t}\begin{bmatrix} -\frac{\sqrt{2}}{2}\cos\sqrt{2}t \\ -\sin\sqrt{2}t + \frac{\sqrt{2}}{2}\cos\sqrt{2}t \\ \sin\sqrt{2}t\end{bmatrix}$$

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② b) continued... $= \vec{X}_1 + i\vec{X}_2 \rightarrow$

$$\vec{x} = c_1\vec{b}_1 + a\vec{X}_1 + b\vec{X}_2$$

$$= c_1\begin{bmatrix}1/3\\1/2\\1\end{bmatrix} + ae^{-3t}\begin{bmatrix}\sqrt{2}/2\sin\sqrt{2}t\\ -\cos\sqrt{2}t - \sqrt{2}/2\sin\sqrt{2}t\\ \cos\sqrt{2}t\end{bmatrix}$$

$$+be^{-3t}\begin{bmatrix}-\sqrt{2}/2\cos\sqrt{2}t\\ -\sin\sqrt{2}t + \sqrt{2}/2\cos\sqrt{2}t\\ \sin\sqrt{2}t\end{bmatrix}$$

general soln:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}c_1/3 + e^{-3t}\left(\frac{a\sqrt{2}}{2}\sin\sqrt{2}t - \frac{b\sqrt{2}}{2}\cos\sqrt{2}t\right)\\ c_1/2 + e^{-3t}\left((-a + \frac{\sqrt{2}}{2}b)\cos\sqrt{2}t - (\frac{\sqrt{2}}{2}a + b)\sin\sqrt{2}t\right)\\ c_1 + e^{-3t}\left(a\cos\sqrt{2}t + b\sin\sqrt{2}t\right)\end{bmatrix}$$

$$\begin{bmatrix}x_1(0)\\x_2(0)\\x_3(0)\end{bmatrix} = \begin{bmatrix}c_1/3 & -\frac{b\sqrt{2}}{2}\\ c_1/2 & -a + \frac{\sqrt{2}}{2}b\\ c_1 & a\end{bmatrix} = \underbrace{\begin{bmatrix}1/3 & 0 & -\sqrt{2}/2\\ 1/2 & -1 & \sqrt{2}/2\\ 1 & 1 & 0\end{bmatrix}}_{B}\begin{bmatrix}c_1\\a\\b\end{bmatrix} = \begin{bmatrix}11\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\a\\b\end{bmatrix} = B^{-1}\begin{bmatrix}11\\0\\0\end{bmatrix} \overset{\text{Maple}}{\underset{\downarrow}{=}} \frac{1}{11}\begin{bmatrix}6 & 6 & 6\\ -6 & -6 & 5\\ -9\sqrt{2} & 2\sqrt{2} & 2\sqrt{2}\end{bmatrix}\begin{bmatrix}11\\0\\0\end{bmatrix} = \begin{bmatrix}6\\-6\\-9\sqrt{2}\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2 + e^{-3t}(-3\sqrt{2}\sin\sqrt{2}t + 9\cos\sqrt{2}t)\\ +3 + e^{-3t}(-3\cos\sqrt{2}t + 12\sqrt{2}\sin\sqrt{2}t)\\ 6 + e^{-3t}(-6\cos\sqrt{2}t - 9\sqrt{2}\sin\sqrt{2}t)\end{bmatrix}$$ ivp soln

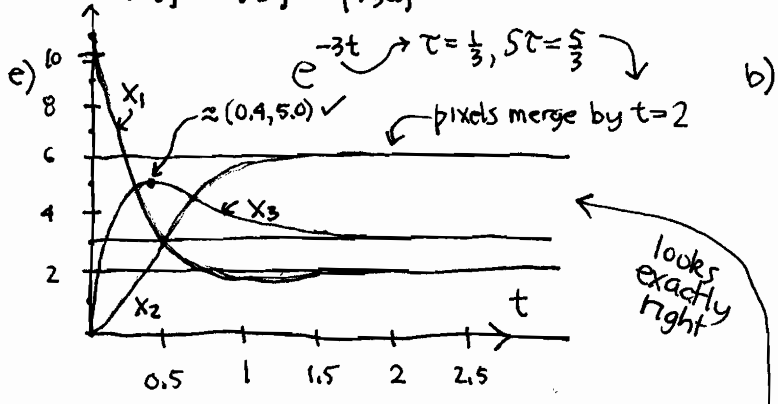
$$-(-6) + \frac{\sqrt{2}}{2}(-9\sqrt{2}) = 6 - 9 = -3$$

$$-\frac{\sqrt{2}}{2}(-6) - (-9\sqrt{2}) = (3+9)\sqrt{2} = 12\sqrt{2}$$

agrees with Maple!

d) $\lim_{t\to 0}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2\\3\\6\end{bmatrix} = \begin{bmatrix}x_{1\infty}\\x_{2\infty}\\x_{3\infty}\end{bmatrix}$

e)



$e^{-3t} \rightarrow \tau = \frac{1}{3}, 5\tau = \frac{5}{3}$

f) $0 = x_2' = -3e^{-3t}(-3\cos\sqrt{2}t + 12\sqrt{2}\sin\sqrt{2}t)$

$+ e^{-3t}(3\sqrt{2}\sin\sqrt{2}t + 24\cos\sqrt{2}t)$

$= e^{-3t}(\underbrace{(9+24)}_{33}\cos\sqrt{2}t + \underbrace{(3-36)}_{-33}\sqrt{2}\sin\sqrt{2}t)$

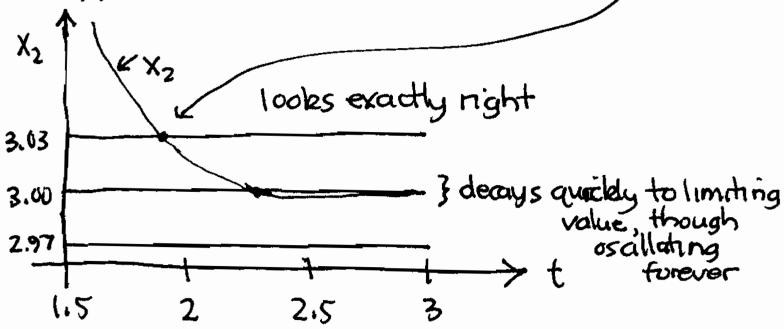
$= 33e^{-3t}(\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t)$

$\rightarrow \cos\sqrt{2}t = \sqrt{2}\sin\sqrt{2}t \rightarrow \tan\sqrt{2}t = \frac{1}{\sqrt{2}} \rightarrow t = \frac{1}{\sqrt{2}}\arctan\frac{1}{\sqrt{2}} \approx 0.435$

$x_2(0.43520) \approx 4.991$

g) $x_{2\infty} = 3$, $1.01(3) = 3.03 = x_2(t) \xrightarrow{\text{Maple}} t \approx 1.924$

$.99(3) = 2.97$



③ a) $\begin{bmatrix}x_1\\x_2\end{bmatrix}' = \underbrace{\begin{bmatrix}-5 & 1\\ 2 & -4\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix}-5-\lambda & 1\\ 2 & -4-\lambda\end{vmatrix} = (\lambda+5)(\lambda+4) - 2$$

$$= \lambda^2 + 9\lambda + 18 = 0$$

$$\lambda = \frac{-9 \pm \sqrt{9^2 - 4\cdot 2\cdot 9}}{2} = \frac{-9 \pm \sqrt{9(9-8)}}{2}$$

$$= \frac{-9 \pm 3}{2} = -3, -6 \text{ (decreasing order)}$$

$$= \lambda_1, \lambda_2$$

$\lambda = -3$:

$$A + 3I = \begin{bmatrix}-5+3 & 1\\ 2 & -4+3\end{bmatrix} = \begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix} \rightarrow \begin{array}{cc}L & F\\ \hline \end{array}\begin{bmatrix}1 & -1/2\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$x_2 = t$, $x_1 = \frac{1}{2}t$ $\langle x_1, x_2\rangle = t\langle\frac{1}{2}, 1\rangle$ $\vec{b}_1$

$\lambda = -6$:

$$A + 6I = \begin{bmatrix}-5+6 & 1\\ 2 & -4+6\end{bmatrix} = \begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix} \rightarrow \begin{array}{cc}L & F\\ \hline \end{array}\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

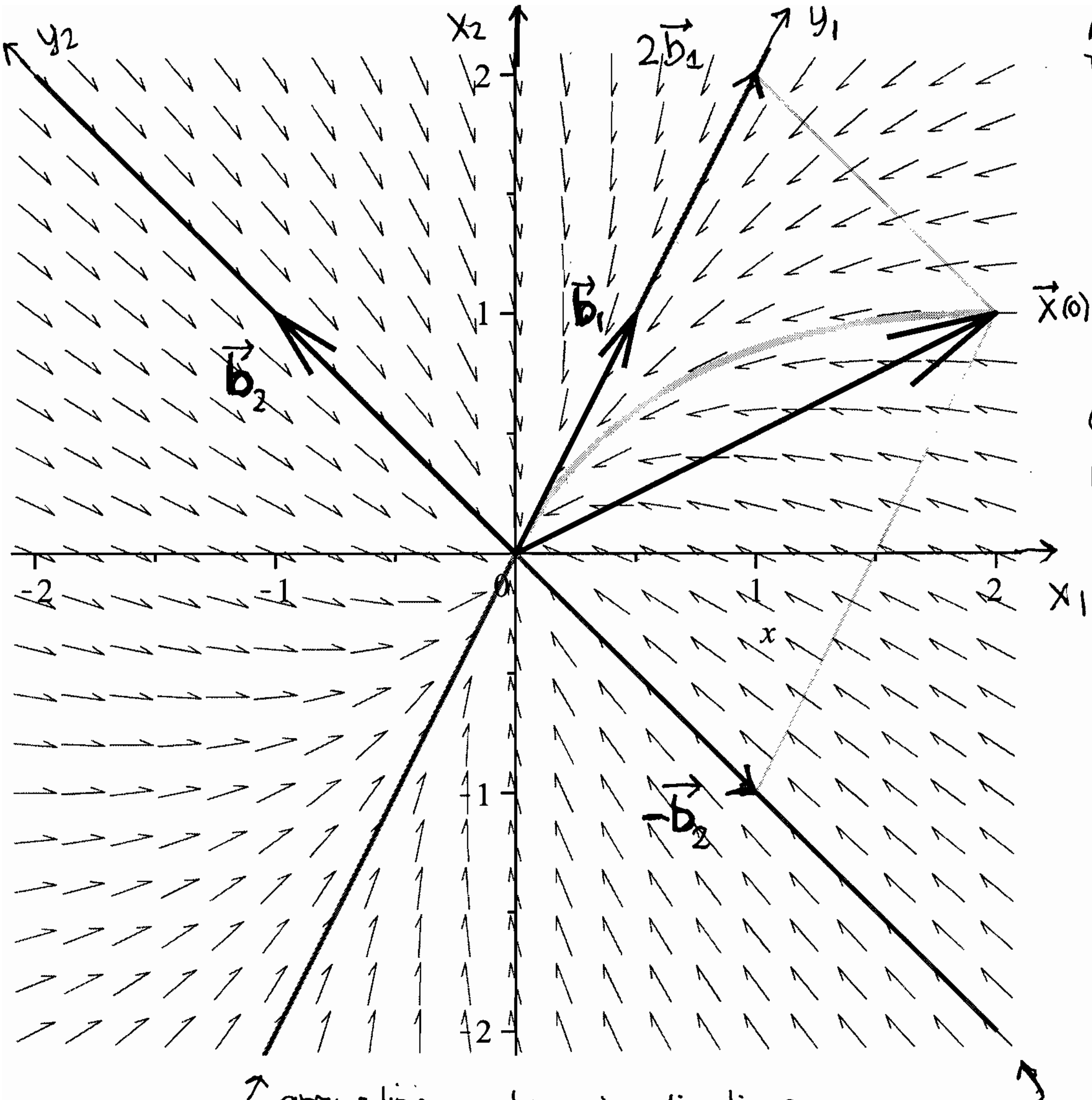
$x_2 = t$, $x_1 = -t$ $\langle x_1, x_2\rangle = t\langle -1, 1\rangle$ $\vec{b}_2$

$$B = \langle\vec{b}_1|\vec{b}_2\rangle = \begin{bmatrix}1/2 & -1\\ 1 & 1\end{bmatrix} \quad B^{-1} = \frac{1}{3/2}\begin{bmatrix}1 & 1\\ -1 & 1/2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2 & 2\\ -2 & 1\end{bmatrix}$$

b) $\vec{x} = B\vec{y}$, $\vec{y} = B^{-1}\vec{x}$

$$\vec{x} = \begin{bmatrix}2\\1\end{bmatrix} \rightarrow \vec{y} = \frac{1}{3}\begin{bmatrix}2 & 2\\ -2 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}6\\-3\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$$

③ c) plot



$\vec{x}(0) = \langle 2,1 \rangle$
$= 2\vec{b}_1 - 1\vec{b}_2$
(exactly right)
(see sides of parallelogram)

arrows line up along eigendirections as they should — arrows point towards origin indicating negative eigenvalues