

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses and root finding.** *Print* requested technology plots, annotate them appropriately by hand and attach to the relevant problems.

1. The displacement $y(t)$ of an underdamped harmonic oscillator system satisfies

$$m y''(t) + c y'(t) + K y(t) = F(t).$$

Let $m = 1$, $c = 2$, $K = 26$ and the initial conditions $y(0) = 0$, $y'(0) = 0$.

a) What are the natural frequency ω_0 , natural decay time τ_0 , the quality factor $Q = \omega_0 \tau_0$ and the period $T_0 = 2\pi/\omega_0$ for this system? (Give both exact and numeric values to 3 decimal places.)

Consider the following driving force functions $F(t)$:

b) $F(t) = 0$.

Find the general solution of the differential equation. What is the frequency ω_1 , decay time τ_1 and the period

$T_1 = 2\pi/\omega_1$ for this decaying sinusoidal solution? (Give both exact and numeric values to 3 decimal places.)

c) $F(t) = 1$.

Find the initial value problem solution by hand and evaluate the asymptotic value $y_\infty = \lim_{t \rightarrow \infty} y(t)$.

Make a plot in an appropriate viewing window ($t \geq 0$ up to 5 decay times) showing both the solution and its horizontal asymptote together, showing clearly the initial oscillating behavior and the approach to the asymptotic value. Does the period of the damped oscillation in your plot roughly agree with T_1 ? Support your claim.

d) $F(t) = \cos(5t)$.

Find the initial value problem solution by hand (but check with Maple!)

Evaluate the values of the amplitude A and phase shift δ for the steady state solution (the part of the solution which remains after the transient has died away) and express the phase shift in radians, degrees and cycles. Make a single plot in an appropriate viewing window showing both the solution function and the steady state solution until they merge, and for comparison with the driving cosine, plot also $A \cos(5t)$ (same amplitude as the steady state solution) to see how the peaks of the steady state solution compare to the peaks of the driving function. Does your plot agree with the phase shift you calculated in fractions of a cycle?

e) $F(t) = \cos(\omega t)$.

Explore resonance for this system by finding the steady state solution by hand, where the nonnegative frequency ω of the driving force function is a parameter.

f) Evaluate the steady state amplitude function $A(\omega)$ and use calculus to find the exact and numerical value of the frequency ω_p and the amplitude $A(\omega_p)$ where it has its peak value for $\omega \geq 0$.

What is the numerical value of the ratio $A(\omega_p)/A(0)$? How does it compare to the quality factor Q ? Does the value of your steady state amplitude $A(0)$ agree with y_∞ from part c) [since $\cos(0t) = 1$] and does $A(5)$ agree with A from part d)?

g) Plot this amplitude function $A(\omega)$ in an appropriate window (showing the behavior of the entire function for $\omega \geq 0$) together with the constant functions $A(\omega_0)$, $A(\omega_p)$ and $A(0)$ and hand annotate on your axes the values of these frequencies and amplitudes and indicate the points on the curve where these values are obtained.

2. $x_1'(t) = -3x_1(t) + x_3(t)$, $x_2'(t) = 3x_1(t) - 2x_2(t)$, $x_3'(t) = 2x_2(t) - x_3(t)$,

$x_1(0) = 11$, $x_2(0) = 0$, $x_3(0) = 0$ describes an closed 3 tank mixing problem.

a) Write this system AND its initial conditions in matrix form, i.e., for the vector variable $\vec{x} = \langle x_1, x_2, x_3 \rangle$ (use arrow notation for vectors!).

b) Use the eigenvector approach to find its general solution, showing all steps.

[Note: to compare your complex eigenvectors with Maple's matrix B of eigenvectors as a check, you need to do $> \text{map}(\text{evalc}, B)$ in order to divide through the quotients of complex numbers into their real and imaginary parts.]

- c) Find the IVP solution, using matrix methods showing all steps. Make sure it agrees with Maple's solution.
- d) Evaluate the three limiting (equilibrium) values: $x_{i\infty} = \lim_{t \rightarrow \infty} x_i$.
- e) Make a single plot showing the three solution curves and their horizontal asymptotes (corresponding to their limiting values) versus t on the same axes for at least 5 characteristic times of the slowest decaying exponential term in these expressions (i.e., in an appropriate viewing window). Hand annotate each curve with its variable name.
- f) The second variable rises from zero and overshoots its asymptotic value, then falling back to it after a maximum. Use calculus to find the maximum value of x_2 for $t \geq 0$ exactly (rules of exponents) and numerically giving both t and x_2 to 3 significant digits. Label this maximum point on the plot with its coordinates. Are your results consistent with your plot?
- g) Find the 3 decimal place numerical value of t (after the maximum!) where x_2 first arrives to within one percent of its limiting value (i.e., equals 1.01 times the limiting value $x_{2\infty}$). Make a close up plot of $x_2, x_{2\infty}, 1.01 x_{2\infty}, 0.99 x_{2\infty}$ showing the variable entering the 1 percent strip around the limiting value, showing a horizontal window starting just to the left of this region and showing the merging of the pixels of x_2 with its limiting value line. Label the 4 curves.
- Does your calculated value of t seem consistent with your plot? Explain.

3. $x_1'(t) = -5x_1(t) + x_2(t), x_2'(t) = 2x_1(t) - 4x_2(t), x_1(0) = 2, x_2(0) = 1$.

- a) Identify the coefficient matrix and find a new basis for R^2 consisting of eigenvectors \vec{b}_1, \vec{b}_2 of this matrix using the standard hand recipe. Order the real eigenvalues $\lambda_1 \geq \lambda_2$ by decreasing value.
- b) Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of the point $\langle x_1, x_2 \rangle = \langle 2, 1 \rangle$ with respect to this basis of eigenvectors.
- c) Use technology to plot a directionfield for this DE with the solution curve through the single initial data point (the DEplot option "dirgrid=[21,21]" is useful), and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose the window $x_1 = -2 \dots 2, x_2 = -2 \dots 2$. By hand label these lines by their coordinate labels, draw in and label the eigenvectors and the initial data vector themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., the parallelogram parallel to the new coordinate axes with the initial data vector as the main diagonal. Do the projections along the coordinate axes agree with the values you found for the new coordinates of this vector? Explain. Does your directionfield correspond to the eigenvectors you have drawn? Explain.

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress me as though it were material for a job interview (you're fired! or you're hired! ?). In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

► solution

▼ pledge

When you have completed the exam, please read and sign the dr bob integrity pledge and hand this test sheet stapled on top of your answer sheets as a cover page, with the first test page facing up:

"During this examination, all work has been my own. I have read the long instructions on the class web page. I give my word that I have not resorted to any ethically questionable means of improving my grade or anyone else's on this examination and that I have not discussed this exam with anyone other than my instructor, nor will I until after the exam period is terminated for all participants."

Signature:

Date: