MAT2705-01/04 07S Take home Test 3

① $\underbrace{y''}_{1} + \underbrace{10}_{k_0} y' + \underbrace{650}_{\omega_0^2} y = F(t)$

d) $\tau_0 = 1/k_0 = 1/10$, $\omega_0 = \sqrt{650} = 5\sqrt{26} \approx 25.50$

$Q = \omega_0 \tau_0 = \frac{5\sqrt{26}}{10} = \frac{\sqrt{26}}{2} \approx 2.55 > \frac{1}{2}$ underdamped

homogeneous soln:

$y = e^{rt}$: $\underbrace{(r^2 + 10r + 650)}_{=0} e^{rt} = 0$

$r = -5 \pm i25$, $e^{rt} = e^{-5t} e^{\pm 25i} = e^{-5t}(\cos 25t \pm i \sin 25t)$

real solution: $y_h = e^{-5t}(c_1 \cos 25t + c_2 \sin 25t)$

a) $F(t) = 100$ $D(100) = 0 \to r = 0$, $e^{t0} = 1$

$y_p = c_3$, $y_p' = 0$, $y_p'' = 0 \to 0 + 0 + 650 c_3 = 100 \to c_3 = \frac{100}{650} = \frac{2}{13}$

$y = e^{-5t}(c_1 \cos 25t + c_2 \sin 25t) + 2/13$

$y' = -5e^{-5t}(c_1 \cos 25t + c_2 \sin 25t) + e^{-5t}(-25 c_1 \sin 25t + 25 c_2 \cos 25t)$

$y(0) = c_1 + \frac{2}{13} = 0 \to c_1 = -2/13$

$y'(0) = -5c_1 + 25 c_2 = 0 \to c_2 = \frac{5c_1}{25} = \frac{1}{5}\left(-\frac{2}{13}\right) = -\frac{2}{65}$

$$\boxed{y = e^{-5t}\left(-\frac{2}{13}\cos 25t - \frac{2}{65}\sin 25t\right) + \frac{2}{13}}$$

$\boxed{\lim_{t\to\infty} y = \frac{2}{13}} \approx 0.154$

[see Maple worksheet for plots]

b) $F(t) = 100 e^{-\frac{t}{5}} \to r = -\frac{1}{5}$ $(D + \frac{1}{5})F(t) = 0$

650 [$y_p = c_3 e^{-t/5}$]

10 [$y_p' = -\frac{1}{5} c_3 e^{-t/5}$]

1 [$y_p'' = \frac{1}{25} c_3 e^{-t/5}$]

$y_p'' + 10y_p' + 650 y_p = (650 - 2 + \frac{1}{25}) c_3 e^{-t/5} = 100 e^{-t/5}$

$(648 + \frac{1}{25}) c_3 = 100 \to c_3 = \frac{2500}{16201} \approx 0.1543$

$y = e^{-5t}(c_1 \cos 25t + c_2 \sin 25t) + c_3 e^{-t/5}$

$y' = -5e^{-5t}(c_1 \cos 25t + c_2 \sin 25t) - \frac{c_3}{5} e^{-t/5}$
$+ e^{-5t}(-25 c_1 \sin 25t + 25 c_2 \cos 25t)$

$y(0) = c_1 + c_3 = 0 \longrightarrow$ $c_1 = -c_3 = -\frac{2500}{16201} \approx -0.1543$

$y'(0) = -5c_1 + 25 c_2 - \frac{c_3}{5} = 0$ $c_2 = \frac{5c_1}{25} + \frac{c_3}{125} \overset{\text{maple}}{=} -\frac{480}{16201} \approx -0.02963$

$$\boxed{y = \frac{-10}{16201} e^{-5t}(250 \cos 25t + 48 \sin 25t) + \frac{2500}{16201} e^{-t/5}}$$

obviously $\lim_{t\to\infty} y = 0$

c) $F(t) = 100 \cos 25t \to r = \pm 25i$ $(D^2 + 25^2) F(t) = 0$

650 [$y_p = c_3 \cos 25t + c_4 \sin 25t$]

10 [$y_p' = -25 c_3 \sin 25t + 25 c_4 \cos 25t$]

1 [$y_p'' = -25^2 c_3 \cos 25t - 25^2 c_4 \sin 25t$]

$y_p'' + 10y_p' + 650 y_p = [(650 - 625) c_3 + 250 c_4] \cos 25t = 100 \cos 25t$
$+ [-250 c_3 + (650 - 625) c_4] \sin 25t$

①c) $\begin{bmatrix} 25 & 250 \\ -250 & 25 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$ $\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{25^2 + 25^2 \cdot 100} \begin{bmatrix} 25 & -250 \\ 250 & 25 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$

$= \frac{1}{25^2 \, 101} \begin{bmatrix} 25 \cdot 100 \\ 250 \cdot 100 \end{bmatrix} = \begin{bmatrix} 4/101 \\ 40/101 \end{bmatrix}$

$y = y_h + \frac{4}{101}(\cos 25t + 10 \sin 25t)$

$y' = y_h' + \frac{4}{101}(-25 \sin 25t + 250 \cos 25t)$

$y(0) = c_1 + \frac{4}{101} \to c_1 = -\frac{4}{101}$

$y'(0) = -5c_1 + 25 c_2 + 4 \cdot \frac{250}{101} \to c_2 \overset{\text{Maple}}{=} -\frac{204}{505}$

$$\boxed{y = -\frac{4}{101} e^{-5t}\left[\cos 25t + \frac{51}{5}\sin 25t\right] + \frac{4}{101}\left[\cos 25t + 10 \sin 25t\right]}$$

e) 650 [$y_p = c_3 \cos \omega t + c_4 \sin \omega t$]

10 [$y_p' = -\omega c_3 \sin \omega t + \omega c_4 \cos \omega t$]

1 [$y_p'' = -\omega^2 c_3 \cos \omega t - \omega^2 c_4 \sin \omega t$]

$y_p'' + 10y_p' + 650 y_p = [c_3(650 - \omega^2) + 10\omega c_4] \cos \omega t$
$+ [-10\omega c_3 + (650 - \omega^2) c_4] \sin \omega t$
$= 100 \cos \omega t$

$\begin{bmatrix} 650 - \omega^2 & 10\omega \\ -10\omega & 650 - \omega^2 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$

$\begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \frac{1}{(650 - \omega^2)^2 + 100\omega^2} \begin{bmatrix} 650 - \omega^2 & -10\omega \\ 10\omega & 650 - \omega^2 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \end{bmatrix}$

$= \frac{100}{(650 - \omega^2)^2 + 100\omega^2} \begin{bmatrix} 650 - \omega^2 \\ 10\omega \end{bmatrix}$

$A = \sqrt{c_3^2 + c_4^2} = \frac{100}{(650^2 - \omega^2)^2 + 100\omega^2} \sqrt{(650 - \omega^2)^2 + 100\omega^2}$

$= \frac{100}{\sqrt{(650 - \omega^2) + 100\omega^2}}$ amplitude of steady state solution

$y_p = \frac{(650 - \omega^2) \cos \omega t + 10\omega \sin \omega t}{(650 - \omega^2)^2 + 100\omega^2}$

f) $A(\omega) = \frac{100}{\sqrt{(650 - \omega^2)^2 + 100\omega^2}}$

$A(0) = \frac{100}{650} = \frac{2}{13} \approx 0.154$

g) $0 = A'(\omega) = 100(-\frac{1}{2})[\ldots]^{-3/2}[2(650 - \omega^2)(-2\omega) + 100(2\omega)]$

$-2(650 - \omega^2) + 100 = 0 \to \omega^2 = 650 - 50 = 600$

$\omega_p = 10\sqrt{6} \approx 24.495 \lessapprox \omega_0 = 25$

$A(\omega_p) = \frac{100}{\sqrt{(650 - 600)^2 + 100 \cdot 600}} = \frac{100}{\sqrt{25 \cdot 10^2 + 6 \cdot 100^2}} = \frac{2}{5} = 0.4$

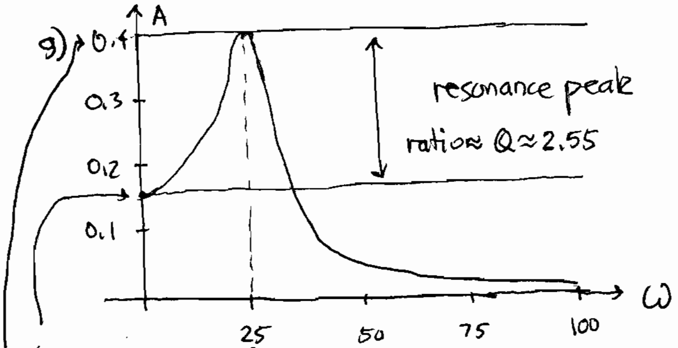
$\frac{A(\omega_p)}{A(0)} = \frac{2/5}{2/13} = \frac{13}{5} = 2.6 \approx Q \approx 2.55$ very close

① f) $A(25) = \dfrac{100}{\sqrt{(650-625)^2 + 100\cdot 625}} = \dfrac{100}{\sqrt{101\cdot 25^2}} = \dfrac{4}{\sqrt{101}}$

part c) $y_p = \frac{4}{101}(\cos 25t + 10\sin 25t)$

$A = \frac{4}{101}\sqrt{1+10^2} = \frac{4}{\sqrt{101}}$ ✓

g)



$A(0) \approx 0.154$ $\quad$ $\omega_p \approx \omega_0 = 25$

$A(\omega_p) \approx 0.4 \approx A(\omega_0)$

(for remaining plots, see worksheet)

②

a) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}' = \begin{bmatrix}-5&6&6\\1&-4&-2\\-3&6&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \leftrightarrow \vec{x}' = A\vec{x},\ \vec{x}(0) = \begin{bmatrix}0\\0\\1\end{bmatrix}$

b) $|A-\lambda I| = \begin{vmatrix}-5-\lambda&6&6\\1&-4-\lambda&-2\\-3&6&4-\lambda\end{vmatrix} \overset{\text{maple}}{=} -\lambda^3-5\lambda^2-8\lambda-4=0$

$\hookrightarrow \lambda = -1,-2,-2$

$\lambda=-1$:

$A+I = \begin{bmatrix}-4&6&6\\1&-3&-2\\-3&6&5\end{bmatrix} \xrightarrow[\text{maple}]{\text{rref}} \begin{array}{ccc}L&L&F\\ \hline \end{array}\begin{bmatrix}1&0&-1\\0&1&1/3\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$x_1 - x_3 = 0$ $\quad$ $x_1 = t$

$x_2 + x_3/3 = 0$ $\quad$ $x_2 = -t/3$ $\quad$ $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = t\underbrace{\begin{bmatrix}1\\-1/3\\1\end{bmatrix}}_{\vec{b}_1}$

$x_3 = t$

$\lambda=-2$:

$A+2I = \begin{bmatrix}-3&6&6\\1&-2&-2\\-3&6&6\end{bmatrix} \xrightarrow[\text{maple}]{\text{rref}} \begin{array}{ccc}L&F&F\end{array}\begin{bmatrix}1&-2&-2\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$x_1 - 2x_2 - 2x_3 = 0 \to x_1 = 2t_1 + 2t_2$ $\quad$ $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}2t_1+2t_2\\t_1\\t_2\end{bmatrix} = t_1\underbrace{\begin{bmatrix}2\\1\\0\end{bmatrix}}_{\vec{b}_2} + t_2\underbrace{\begin{bmatrix}2\\0\\1\end{bmatrix}}_{\vec{b}_3}$

$x_2 = t_1$

$x_3 = t_2$

maple Eigenvectors exchanges these →

$\lambda = \quad -1 \quad -2 \quad -2$

$B = \langle \vec{b}_1|\vec{b}_2|\vec{b}_3\rangle = \begin{bmatrix}1&2&2\\-1/3&1&0\\1&0&1\end{bmatrix} \to A_B = B^{-1}AB \overset{\text{maple}}{=} \begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-2\end{bmatrix}$ diagonalized

$B^{-1} \overset{\text{maple}}{=} \begin{bmatrix}-3&6&6\\-1&3&2\\3&-6&-5\end{bmatrix}$

② b) $\vec{x}' = A\vec{x}$ $\qquad$ $B^{-1}[(B\vec{y})' = A(B\vec{y})]$

$\vec{x} = B\vec{y},\ \vec{y} = B^{-1}\vec{x}$ $\qquad$ $\vec{y}' = B^{-1}AB\vec{y} = A_B\vec{y}$

$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}' = \begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-2\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}-y_1\\-2y_2\\-2y_3\end{bmatrix}$ $\quad$ $\begin{array}{ll} y_1' = -y_1 & y_1 = c_1e^{-t}\\ y_2' = -2y_2 & y_2 = c_2e^{-2t}\\ y_3' = -2y_3 & y_3 = c_3e^{-2t}\end{array}$

$\vec{x} = B\vec{y} = \begin{bmatrix}1&2&2\\-1/3&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}c_1e^{-t}\\c_2e^{-2t}\\c_3e^{-2t}\end{bmatrix} = \begin{bmatrix}c_1e^{-t} + (2c_2+2c_3)e^{-2t}\\ -\frac{1}{3}c_1e^{-t} + c_2e^{-2t}\\ c_1e^{-t} + c_3e^{-2t}\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

general solution

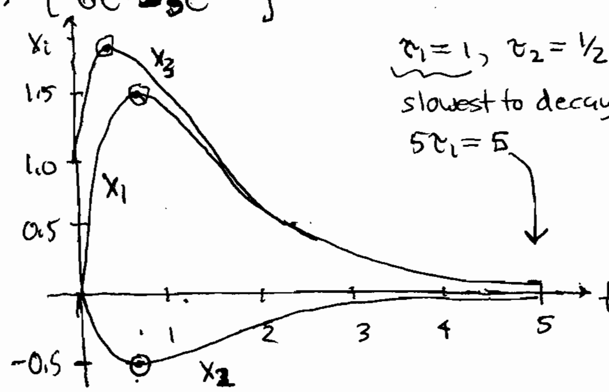
c) $t=0$:

$\underbrace{\begin{bmatrix}1&2&2\\-1/3&1&0\\1&0&1\end{bmatrix}}_{B}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = B^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-3&6&6\\-1&3&2\\3&-6&-5\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}$

$= \begin{bmatrix}6\\2\\-5\end{bmatrix}$ $\qquad$ $2c_2 + 2c_3 = 2(-3) = -6$

$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}6e^{-t} - 6e^{-2t}\\ -2e^{-t} + 2e^{-2t}\\ 6e^{-t} - 5e^{-2t}\end{bmatrix}$ IVP solution

d)



$\tau_1 = 1,\ \tau_2 = 1/2 = \tau_3$

slowest to decay

$5\tau_1 = 5$

e) $x_1 = 6e^{-t} - 6e^{-2t}$

$x_1' = -6e^{-t} + 12e^{-2t} = 0 \to [e^{-t} = 2e^{-2t}]e^{2t}$

$e^t = 2$ $\quad$ $\boxed{t_1 = \ln 2 \approx 0.693}$

$x_2 = -2e^{-t} + 2e^{-2t}$, $x_2' = 2e^{-t} - 4e^{-2t} = 0$

$[e^{-t} = 2e^{-2t}]e^{2t} \to e^t = 2 \to \boxed{t_2 = \ln 2 \approx 0.693}$

$x_3 = 6e^{-t} - 5e^{-2t}$, $x_3' = -6e^{-t} + 10e^{-2t}$

$[e^{-t} = \frac{5}{3}e^{-2t}]e^{2t}$ $\quad$ $e^t = \frac{5}{3}$ $\quad$ $\boxed{t_3 = \ln 5/3 \approx 0.511}$

$x_1(t_1) = 6e^{-\ln 2} - 6e^{-2\ln 2} = 6\left(\frac{1}{2} - \frac{1}{4}\right) = \boxed{\frac{3}{2} = 1.5}$

$x_2(t_2) = 2(-e^{-\ln 2} + e^{-2\ln 2}) = 2\left(-\frac{1}{2} + \frac{1}{4}\right) = \boxed{-\frac{1}{2} = -0.5}$

$x_3(t_3) = 6e^{-\ln\frac{5}{3}} - 5e^{-2\ln\frac{5}{3}} = 6\left(\frac{3}{5}\right) - 5\left(\frac{9}{25}\right) = \boxed{\frac{9}{5} = 1.8}$

The 3 points (0.69, 1.5), (0.69, −0.5), (0.51, 1.8) are right on the money (see circled points).

f) $x_1(t) = .01(1.5) = 0.015 \to t \approx 5.989$

$x_2(t) = .01(-0.5) = -.005 \to t \approx 5.989$

$x_3(t) = .01(1.8) = 0.018 \to t \approx 5.801$

numerical soln makes more sense here even tho these conditions are quadratic in e^{-t} & can be solved exactly. (maple used)

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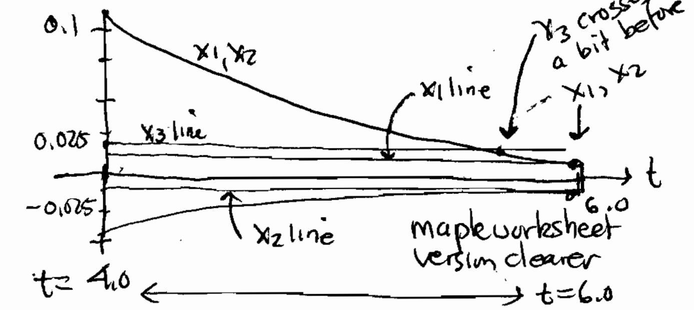
② f) you need a closeup to see where the 1% values are crossed



actually $t = 5..6$ is a better view

③ a) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 1 & -4 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leftrightarrow \vec{x}\,' = A\vec{x},\ \vec{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -4 \\ 2 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) + 8$
$= (\lambda-1)(\lambda+3) + 8 = \lambda^2 + 2\lambda + 5 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-4(5)}}{2} = -1 \pm \sqrt{-4} = -1 \pm 2i$

$\lambda = -1+2i$

$A - \lambda I = \begin{bmatrix} 1-(-1+2i) & -4 \\ 2 & -3-(-1+2i) \end{bmatrix} = \begin{bmatrix} 2-2i & -4 \\ 2 & -2-2i \end{bmatrix}$

rref $\longrightarrow$ swap, divide eliminate R_2

$\begin{array}{cc} L & F \end{array}$
$\begin{bmatrix} 1 & -1-i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad x_1 - (1+i)x_2 = 0$
$x_2 = t$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \underbrace{\begin{bmatrix} 1+i \\ 1 \end{bmatrix}}_{\vec{b}_1} \rightarrow \vec{b}_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix}$

$A_B = B^{-1}AB = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix}$

b)
$B^{-1}(\vec{x}\,' = A\vec{x}) \rightarrow \vec{y}\,' = A_B\vec{y},\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -1+2i & 0 \\ 0 & -1-2i \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$x = By,\ y = B^{-1}x$

$y_1' = (-1+2i)y_1,\quad y_1 = c_1 e^{(-1+2i)t} = c_1 e^{-t}(\cos 2t + i\sin 2t)$
$y_2' = (-1-2i)y_2,\quad y_2 = c_2 e^{(-1-2i)t} = c_2 e^{-t}(\sin 2t - i\sin 2t)$

$\vec{x} = B\vec{y} = y_1\vec{b}_1 + y_2\vec{b}_2 = c_1 e^{(-1+2i)t}\vec{b}_1 + c_2 e^{(-1-2i)t}\vec{b}_2$

$= e^{-t}(\cos 2t + i\sin 2t)\begin{bmatrix} 1+i \\ 1 \end{bmatrix} = e^{-t}\begin{bmatrix} \cos 2t - \sin 2t + i(\cos 2t + \sin 2t) \\ \cos 2t \quad + i \ \sin 2t \end{bmatrix}$

$= \underbrace{e^{-t}\begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{bmatrix}}\ + i\ \underbrace{e^{-t}\begin{bmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{bmatrix}}$

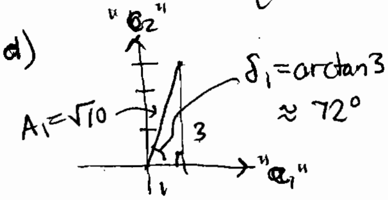
new real basis of solution space so

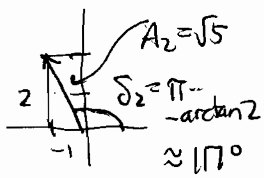
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a e^{-t}\begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{bmatrix} + b e^{-t}\begin{bmatrix} \cos 2t + \sin 2t \\ \sin 2t \end{bmatrix} = \begin{bmatrix} e^{-t}[(a+b)\cos 2t + (-a+b)\sin 2t] \\ e^{-t}[a \ \cos 2t \ + b\ \sin 2t] \end{bmatrix}$ gen soln

c) $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} a+b \\ a \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad a+b = 1 \rightarrow b = 2$
$a = -1$
$b - a = 3$

IVP soln: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{-t}(\cos 2t + 3\sin 2t) \\ e^{-t}(-\cos 2t + 2\sin 2t) \end{bmatrix}$

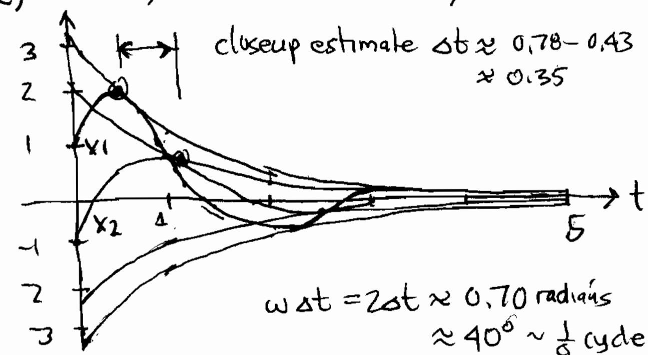
d)





$x_1 = \sqrt{10}\, e^{-t}\cos(2t - \arctan 3)$
$x_2 = \sqrt{5}\, e^{-t}\cos(2t - \pi + \arctan 2)$

e) $\tau = 1,\ 5\tau = 5,\ \sqrt{10} \approx 3.162,\ \sqrt{5} \approx 2.236$



closeup estimate $\Delta t \approx 0.78 - 0.43 \approx 0.35$

$\omega \Delta t = 2\Delta t \approx 0.70$ radians
$\approx 40° \sim \frac{1}{9}$ cycle
close enuf

f) $\delta_1 - \delta_2 = \arctan 3 - (\pi + \arctan 2)$
$\approx -.78540 \approx -45.000° = -\frac{1}{8}$ cycle

x_1 is ahead of x_2 in time — its peaks occur first followed by those of x_2 after 1/8 cycle.