MAT2705-01/06 07F FINAL EXAM ANSWERS (1)

1 a) $x'' + 13x' + 36x = 25\sin(3t)$

$x_p \sim e^{rt} \to r^2 + 13r + 36 = 0$

$r = \dfrac{-13 \pm \sqrt{13^2 - 4\cdot 36}}{2} = \dfrac{-13 \pm \sqrt{169-144}}{2} = \dfrac{-13 \pm \sqrt{25}}{2} = \dfrac{-13 \pm 5}{2}$

$= -\frac{18}{2}, -\frac{8}{2} = -9, -4, \quad e^{rt} = e^{-4t}, e^{-9t}$

$x_h = c_1 e^{-4t} + c_2 e^{-9t}$

$36\,(x_p = c_3\cos 3t + c_4 \sin 3t) \to 36x_p = 36c_3\cos 3t + 36c_4 \sin 3t$

$13\,(x_p' = -3c_3 \sin 3t + 3c_4 \cos 3t) \to 13x_p' = 39c_4\cos 3t - 39c_3 \sin 3t$

$1\,(x_p'' = -9c_3\cos 3t - 9c_4 \sin 3t) \to x_p'' = -9c_3 \cos 3t - 9c_4 \sin 3t$

$x_p'' + 13x_p' + 36x_p = \underbrace{(27c_3 + 39c_4)}_{=0}\cos 3t + \underbrace{(-39c_3 + 27c_4)}_{=25}\sin 3t = 25\sin 3t$

$27c_3 + 39c_4 = 0$
$-39c_3 + 27c_4 = 25$

$\begin{bmatrix}27 & 39\\ -39 & 27\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \begin{bmatrix}0\\ 25\end{bmatrix} \to$ RREF or use inverse

$\begin{bmatrix}c_3\\ c_4\end{bmatrix} = \frac{1}{2250}\begin{bmatrix}27 & -39\\ 39 & 27\end{bmatrix}\begin{bmatrix}0\\ 25\end{bmatrix} = \frac{1}{90}\begin{bmatrix}-39\\ 27\end{bmatrix} = \begin{bmatrix}-13/30\\ 3/10\end{bmatrix}$

$x_p = -\frac{13}{30}\cos 3t + \frac{3}{10}\sin 3t$

$\boxed{x = c_1 e^{-4t} + c_2 e^{-9t} - \frac{13}{30}\cos 3t + \frac{3}{10}\sin 3t}$

b) $x' = -4c_1 e^{-4t} - 9c_2 e^{-9t} + \frac{13}{10}\sin 3t + \frac{9}{10}\cos 3t$

$x(0) = c_1 + c_2 - 13/30 = 5$
$x'(0) = -4c_1 - 9c_2 + 9/10 = 5$

$\begin{bmatrix}1 & 1\\ -4 & -9\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}5 + 13/30\\ 5 - 9/10\end{bmatrix} \xrightarrow[\text{use inverse}]{\text{RREF or}}$

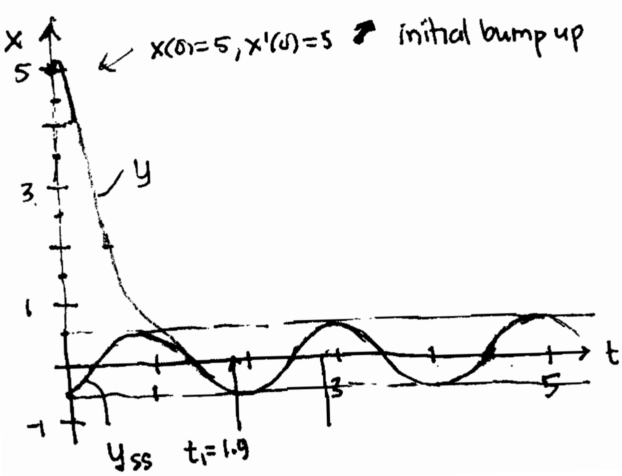
$\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \frac{1}{-5}\begin{bmatrix}-9 & -1\\ 4 & 1\end{bmatrix}\begin{bmatrix}163/30\\ 41/10\end{bmatrix} = -\frac{1}{5}\begin{bmatrix}(-3(163) - 41)/10\\ (4(163) + 41\cdot 3)/30\end{bmatrix} = \begin{bmatrix}53/5\\ -31/6\end{bmatrix}$

$\boxed{x = \frac{53}{5}e^{-4t} - \frac{31}{6}e^{-9t} - \frac{13}{30}\cos 3t + \frac{3}{10}\sin 3t}$

c) $A = \frac{1}{30}\sqrt{13^2 + 9^2} = \frac{\sqrt{250}}{30} = \boxed{\frac{\sqrt{10}}{6} \approx 0.527}$

$\frac{53}{5}e^{-4t} - \frac{31}{6}e^{-9t} = .01\left(\frac{\sqrt{10}}{6}\right) \xrightarrow[\text{solve } t>0]{\text{num}} \boxed{t_1 \approx 1.902}$

$1\% \longrightarrow .01$



2 a) $\begin{bmatrix}x_1\\ x_2\end{bmatrix}'' = \underbrace{\begin{bmatrix}-7 & 1\\ 6 & -6\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{X}, \quad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}5\\ 5\end{bmatrix}, \quad \begin{bmatrix}x_1'(0)\\ x_2'(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$\underline{X}'' = A\underline{X}, \quad \underline{X}(0) = \begin{bmatrix}5\\ 5\end{bmatrix}, \quad \underline{X}'(0) = \begin{bmatrix}0\\ 0\end{bmatrix}$ or

$\vec{x}'' = A\vec{x}, \quad \vec{x}(0) = \begin{bmatrix}5\\ 5\end{bmatrix}, \quad \vec{x}'(0) = \begin{bmatrix}0\\ 0\end{bmatrix}$

b) $|A - \lambda I| = \begin{vmatrix}-7-\lambda & 1\\ 6 & -6-\lambda\end{vmatrix} = (\lambda+7)(\lambda+6) - 6 = \lambda^2 + 13\lambda + 42 - 6$

$= \lambda^2 + 13\lambda + 36 = 0$

$\lambda = -4, -9$ as before

$\lambda = -4$:

$A + 4I = \begin{bmatrix}-7+4 & 1\\ 6 & -6+4\end{bmatrix} = \begin{bmatrix}-3 & 1\\ 6 & -2\end{bmatrix}$

$\xrightarrow{\text{RREF}} \begin{array}{c}\begin{array}{cc}L & F\end{array}\\ \begin{bmatrix}1 & -1/3\\ 0 & 0\end{bmatrix}\end{array}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$x_1 - \frac{1}{3}x_1 = 0 \to x_1 = t/3, \quad \begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}t/3\\ t\end{bmatrix} = t\underbrace{\begin{bmatrix}1/3\\ 1\end{bmatrix}}_{\vec{b}_1}$

$x_2 = t$

$\lambda = -9$:

$A + 9I = \begin{bmatrix}-7+9 & 1\\ 6 & -6+9\end{bmatrix} = \begin{bmatrix}2 & 1\\ 6 & 3\end{bmatrix}$

$\xrightarrow{\text{RREF}} \begin{array}{c}\begin{array}{cc}L & F\end{array}\\ \begin{bmatrix}1 & 1/2\\ 0 & 0\end{bmatrix}\end{array}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$x_1 + \frac{1}{2}x_2 = 0 \to x_1 = -\frac{1}{2}t, \quad \begin{bmatrix}x_1\\ x_2\end{bmatrix} = t\underbrace{\begin{bmatrix}-1/2\\ 1\end{bmatrix}}_{\vec{b}_2}$

$x_2 = t$

$B = \langle b_1 | b_2 \rangle = \begin{bmatrix}1/3 & -1/2\\ 1 & 1\end{bmatrix}$

$B^{-1} = \frac{1}{\frac{1}{3} + \frac{1}{2}}\begin{bmatrix}1 & 1/2\\ -1 & 1/3\end{bmatrix} = \frac{6}{5}\begin{bmatrix}+1 & 1/2\\ -1 & 1/3\end{bmatrix}$

$A_B = B^{-1}AB = \begin{bmatrix}-4 & 0\\ 0 & -9\end{bmatrix}$

$\underline{x} = y_1\underline{b_1} + y_2\underline{b_2} = B\begin{bmatrix}y_1\\ y_2\end{bmatrix} = B\underline{y} \to \underline{y} = B^{-1}\underline{x}$

$\underline{y}(0) = B^{-1}\underline{x}(0) = \frac{6}{5}\begin{bmatrix}1 & 1/2\\ -1 & 1/3\end{bmatrix}\begin{bmatrix}5\\ 5\end{bmatrix} = 6\begin{bmatrix}3/2\\ -2/3\end{bmatrix} = \boxed{\begin{bmatrix}9\\ -4\end{bmatrix}}$

these agree exactly with the parallelogram sides!

c) $A\underline{x} + \underline{f} = \underline{0} \to \underline{x} = -A^{-1}\underline{f}$

$= -\frac{1}{42-6}\begin{bmatrix}-6 & -1\\ -6 & -7\end{bmatrix}\begin{bmatrix}36\\ 36\end{bmatrix} = \begin{bmatrix}6 & +1\\ 6 & +7\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix} = \boxed{\begin{bmatrix}7\\ 13\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}}$

d) $\underline{x}'' = A\underline{x} \to \underline{y}'' = B^{-1}\underline{x}'' = B^{-1}(A\underline{x}) = B^{-1}AB\underline{y} = A_B\underline{y}$

$\begin{bmatrix}y_1\\ y_2\end{bmatrix}'' = \begin{bmatrix}-4 & 0\\ 0 & -9\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}-4y_1\\ -9y_2\end{bmatrix}$

$y_1'' + 4y_1 = 0 \quad y_1 = c_1\cos 2t + c_2 \sin 2t$

$y_2'' + 9y_2 = 0 \quad y_2 = c_3\cos 3t + c_4 \sin 3t$

MAT2705-01/06 07F FINAL EXAM ANSWERS (2)

② d) $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1\cos 2t + c_2 \sin 2t \\ c_3 \cos 3t + c_4 \sin 3t \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(c_1\cos 2t + c_2\sin 2t) - \frac{1}{2}(c_3\cos 3t + c_4 \sin 3t) \\ c_1\cos 2t + c_2 \sin 2t \;+\; c_3\cos 3t + c_4\sin 3t \end{bmatrix}$ gen soln

e) $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2c_1\sin 2t + 2c_2\cos 2t \\ -3c_3 \sin 3t + 3c_4 \cos 3t \end{bmatrix}$

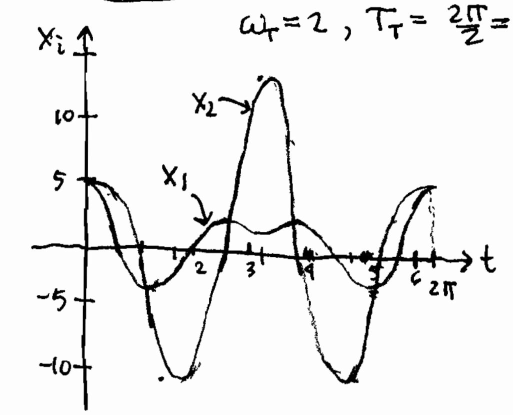
$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_3 \end{bmatrix} = \frac{6}{5}\begin{bmatrix} 1 & 1/2 \\ -1 & 1/3 \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = 6\begin{bmatrix} 3/2 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$ (same as above)

$\begin{bmatrix} x_1'(0) \\ x_2'(0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 2c_2 \\ 3c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{matrix} c_2 = 0 \\ c_4 = 0 \end{matrix}$

$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{9\cos 2t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}}_{\text{tandem}} - \underbrace{4\cos 3t \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}_{\text{accordian}} = \begin{bmatrix} 3\cos 2t + 2\cos 3t \\ 9\cos 2t - 4\sin 3t \end{bmatrix}}$ ivp soln

soln must specify how x_1, x_2 are related to t

f) $\omega_T = 2,\ T_T = \frac{2\pi}{2} = \pi\ ;\ \omega_A = 3,\ T_A = \frac{2\pi}{3}$ common period: $\boxed{2\pi = T} = 2T_T = 3T_A$



g) $\underline{x}' = A\underline{x} \to \underline{y}' = A_B \underline{y} \to \begin{matrix} y_1' = -4y_1, & y_1 = c_1 e^{-4t} \\ y_2' = -9y_2 & y_2 = c_2 e^{-9t} \end{matrix}$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{-4t} \\ c_2 e^{-9t} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}c_1 e^{-4t} - \frac{1}{2}c_2 e^{-9t} \\ c_1 e^{-4t} + c_2 e^{-9t} \end{bmatrix}$ gen soln

$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1/3 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \to \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = B^{-1}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$ (as before)

$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3e^{-4t} + 2e^{-9t} \\ 9e^{-4t} - 4e^{-9t} \end{bmatrix}}$ ivp soln.

2b) ▶ **gridline default plot window**

y_1 has slope 3: (right) over 1, up 3
y_2 has slope −2: left 1, up 2

tickmarks on axes show new components are 9 and −4 respectively