MAT2705-01/06 07F Take Home Test 3 Print Name (Last, First) ____________________________________|__

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers EXACT (but give decimal approximations for interpretation). Indicate where technology is used and what type (Maple, GC, MathCad). **You may use technology for row reductions, matrix inverses and root finding.** Print requested technology plots, annotate them appropriately by hand and attach to the relevant problems.

1. The displacement $y(t)$ of a damped harmonic oscillator system satisfies

$$m\,y''(t) + c\,y'(t) + K\,y(t) = F(t).$$

Let $m = 1$, $c = 4$, $K = 13$. Consider the initial conditions $y(0) = 0$, $y'(0) = 0$ and the following driving force functions $F(t)$:

a) $F(t) = 1$.

Find the initial value problem solution by hand and evaluate the asymptotic value $y_\infty = \lim_{t\to\infty} y(t)$.

b) $F(t) = 1 - e^{-2t}$.

Find the initial value problem solution by hand and evaluate the asymptotic value $y_\infty = \lim_{t\to\infty} y(t)$.

c) Make a plot in appropriate viewing windows ($t \geq 0$) showing both these two solution functions and their horizontal asymptotes together, showing clearly the initial oscillating behavior and the approach to the asymptotic value. Identify the solution curves using the notation "a), b)" .

d) $F(t) = \sin(3\,t)$.

Find the initial value problem solution by hand. Make a single plot in an appropriate viewing window showing both the solution function and the steady state solution until they merge. Evaluate the values of the amplitude and phase shift for the steady state solution (the part of the solution which remains after the transient has died away) and express the phase shift in radians, degrees and cycles.

e) What are the natural frequency ω_0, natural decay time τ_0, and the quality factor $Q = \omega_0\tau_0$ (both exact and numeric values) for this system?

f) $F(t) = \sin(\omega\,t)$.

Explore resonance for this system by finding the steady state solution by hand, where the nonegative frequency ω of the driving force function is a parameter.

g) Evaluate the steady state amplitude function $A(\omega)$ and use calculus to find the exact and numerical value of the frequency ω_p and the amplitude $A(\omega_p)$ where it has its peak value for $\omega \geq 0$.

What is the numerical value of the ratio $A(\omega_p)/A(0)$? Does the value of your steady state amplitude for part d) agree with $A(3)$?

g) Plot this amplitude function $A(\omega)$ in an appropriate window (showing the behavior of the entire function for $\omega \geq 0$) together with the constant functions $A(\omega_0)$, $A(\omega_p)$ and $A(0)$ and hand annotate on your axes the values of these frequencies and amplitudes.

2. $x_1'(t) = -3\,x_1(t)$, $x_2'(t) = 3\,x_1(t) - 2\,x_2(t)$, $x_3'(t) = 2\,x_2(t) - x_3(t)$,
$x_1(0) = 9$, $x_2(0) = 0$, $x_3(0) = 0$ describes an open 3 tank mixing problem.

a) Write this system and its initial conditions in matrix form, i.e., for the vector variable $\mathbf{x} = \langle x_1, x_2, x_3\rangle$ (use arrow notation for vectors!).

b) Use the eigenvector approach to find its general solution, showing all steps.

c) Find the IVP solution, showing all steps.

d) Make a single plot showing the three solution curves versus t on the same axes for at least 5 characteristic times of the slowest decaying exponential term in these expressions (i.e., in an appropriate viewing window).

e) Use calculus to find the maximum values of x_1, x_2, x_3 for $t \geq 0$ exactly (rules of exponents) and numerically giving both t and x_i to 3 significant digits. Are your results consistent with your plot?

f) Find the values of t where each of these functions falls to below one percent of the initial total value 9. Do these values seem consistent with your plot? Explain.

3. $x_1'(t) = x_1(t) - 5\,x_2(t), x_2'(t) = x_1(t) + 3\,x_2(t), x_1(0) = 1, x_2(0) = 0$

a) Write this system and its initial conditions in matrix form, i.e., for the vector variable **x** = $\langle x_1, x_2 \rangle$ (use arrow notation for vectors!).
b) Use the eigenvector approach to find its general solution in explicitly real form, showing all steps, and give formulas for the individual scalar variables.
c) Find the IVP solution, showing all steps, and give formulas for the individual scalar variables.
d) Express the sinusoidal factor in each solution function as a phase-shifted cosine.
e) Using your formulas from part c), make a single $t > 0$ plot for an interval about 4 times the characteristic time of the exponential factor, showing both solution curves and their exponentially growing amplitude envelopes of the oscillations, annotating your paper printout to identify the individual solution curves by name. From the graph, estimate the time interval between the first peak of x_1 and the first peak of x_2 and convert it to an angle by multiplying by the frequency.
f) What is the difference in phase between these peaks in radians, degrees and in a fraction of a cycle, as calculated from the difference of your calculated phase shifts in part d)? Are these consistent with your estimate for the time interval between successive peaks in e) converted into an angle? Explain. Which of the two solution curves is ahead in time (has its peaks first)?

4. $x_1'(t) = -6\,x_1(t) + 4\,x_2(t), x_2'(t) = x_1(t) - 6\,x_2(t), x_1(0) = 0, x_2(0) = 3.$

a) Identify the coefficient matrix and find a new basis for R^2 consisting of eigenvectors $\boldsymbol{b}_1, \boldsymbol{b}_2$ of this matrix using the standard hand recipe. Order the real eigenvalues $\lambda_1 \geq \lambda_2$ by decreasing value.
b) Evaluate the new coordinates $\langle y_1, y_2 \rangle$ of the point $\langle x_1, x_2 \rangle = \langle 0, 3 \rangle$ with respect to this basis of eigenvectors.
c) Use technology to plot a directionfield for this DE with the solution curve through the single initial data point (the DEplot option "dirgrid=[21,21]" is useful), and (by hand if necessary) include the lines through the two eigenvectors representing the two subspaces of eigenvectors. Choose the window $x_1 = -3..3, x_2 = -3..3$. By hand label these lines by their coordinate labels, draw in and label the eigenvectors and the initial data vector themselves as arrows, and include the parallelogram projection of the latter vector onto the new coordinate axes, i.e., the parallelogram parallel to the new coordinate axes with the initial data vector as the main diagonal. Do the projections along the coordinate axes agree with the values you found for the new coordinates of this vector? Explain. Does your directionfield correspond to the eigenvectors you have drawn? Explain.

Advice. When in doubt about how much work to show, show more. Explain using words if it helps. Think of this take-home test as an exercise in "writing intensive" technical expression. Try to impress me as though it were material for a job interview (you're fired! or you're hired! ?). In a real world technical job, you need to be able to write coherent technical reports that other people can follow.

▶ solution

▶ pledge