MAT 2705-01/06 07F TEST 2 ANSWERS

① a)
$$\begin{bmatrix} 3 & 1 & -3 & 11 & 10 \\ 5 & 8 & 2 & -2 & 7 \\ 2 & 5 & 0 & -1 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -3 & 11 & 10 & 0 \\ 5 & 8 & 2 & -2 & 7 & 0 \\ 2 & 5 & 0 & -1 & 14 & 0 \end{bmatrix} \xrightarrow[\text{Maple}]{\text{RREF}} \begin{array}{cccccc} \overbrace{L \; L \; L}^{\text{leading}} & & & \overbrace{F \; F}^{\text{free}} & & \\ x_1 \; x_2 \; x_3 & & & x_4 \; x_5 & & \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 & -3 & 0 \\ 0 & 1 & 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & -2 & -5 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_4 - 3x_5 &= 0 \\ x_2 - x_4 + 4x_5 &= 0 \\ x_3 - 2x_4 - 5x_5 &= 0 \\ x_4 &= t_1 \\ x_5 &= t_2 \end{aligned} \longrightarrow \boxed{\begin{aligned} x_1 &= -2t_1 + 3t_2 \\ x_2 &= t_1 - 4t_2 \\ x_3 &= 2t_1 + 5t_2 \\ x_4 &= t_1 \\ x_5 &= t_2 \end{aligned}}$$

b)
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t_1+3t_2 \\ t_1-4t_2 \\ 2t_1+5t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t_2\begin{bmatrix} 3 \\ -4 \\ 5 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \text{basis: } \left\{ \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

c) These coefficient vectors represent the two independent linear relationships; letting $\underline{C}_i$ be the ith column:

$$-2\underline{C}_1 + \underline{C}_2 + 2\underline{C}_3 + \underline{C}_4 = 0$$
$$3\underline{C}_1 - 4\underline{C}_2 - 5\underline{C}_3 + \underline{C}_5 = 0$$

d)
$$\begin{bmatrix} 3 & 1 & -3 & 11 \\ 5 & 8 & 2 & -2 \\ 2 & 5 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -3 & 11 & 10 \\ 5 & 8 & 2 & -2 & 7 \\ 2 & 5 & 0 & 1 & 14 \end{bmatrix} \xrightarrow[\substack{\text{maple} \\ \text{(same as above)}}]{\text{RREF}} \begin{array}{c} L \; L \; L \; F \\ x_1 \; x_2 \; x_3 \; x_4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 2 & -3 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & -2 & 5 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_4 &= -3 & x_1 &= -3 - 2t \\ x_2 - x_4 &= 4 & x_2 &= 4 + t \\ x_3 - 2x_4 &= 5 & x_3 &= 5 + 2t \\ x_4 &= t \end{aligned}$$

$$\boxed{\underline{C}_5 = (-3-2t)\underline{C}_1 + (4+t)\underline{C}_2 + (5+2t)\underline{C}_3 + t\,\underline{C}_4}$$

② a)
$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \\ 1 & 1 & 4 \end{vmatrix} \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 + R_1}{=}} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{vmatrix} = 1(2)(3) = \boxed{6} \neq 0$$

so the columns of A are linearly independent

b)
$$A^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & -3 \\ 2 & 3 & 1 \\ -2 & 0 & 2 \end{bmatrix} \quad \text{Maple}$$

order of matrix factors matters!

$$\underline{A}^{-1}[\underline{A}\underline{x} = \underline{b}] \rightarrow \underline{x} = \underline{A}^{-1}\underline{b}$$

$$= \frac{1}{6}\begin{bmatrix} 6 & -3 & -3 \\ 2 & 3 & 1 \\ -2 & 0 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6-6+0 \\ 2+6+0 \\ -2+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/3 \\ -1/3 \end{bmatrix}$$

ie $x_1 = 0,\ x_2 = 4/3,\ x_3 = -1/3$

$$\underline{x} = \frac{1}{3}\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$$

$$\underline{A}\,\underline{x} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -2 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}/3 = \frac{1}{3}\begin{bmatrix} 4-1 \\ 4+2 \\ 4-4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \underline{b} \checkmark$$

c)
$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & +1 & -2 \\ 1 & 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 2x_1 \\ -x_1 + x_2 - 2x_3 &= 2x_2 \\ x_1 + x_2 + 4x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ -x_1 - x_2 - 2x_3 &= 0 \\ x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ -1 & -1 & -2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & -1 & -2 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix} \xrightarrow[\text{Maple}]{\text{RREF}} \begin{array}{c} L \; L \; F \\ x_1 \; x_2 \; x_3 \end{array} \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + \tfrac{1}{2}x_3 &= 0 & x_1 &= -\tfrac{1}{2}t \\ x_2 + \tfrac{3}{2}x_3 &= 0 & x_2 &= -\tfrac{3}{2}t \\ x_3 &= t & x_3 &= t \end{aligned} \qquad \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t/2 \\ -3t/2 \\ t \end{bmatrix}}$$

understanding the words of linear algebra systems talk

"system of"
3 linear eqns in 5 variables $x_1, \dots, x_5$
(scalar variables = unknowns)

$$\boxed{\begin{aligned} 3x_1 + x_2 - 3x_3 + 11x_4 + 10x_5 &= 0 \\ 5x_1 + 8x_2 + 2x_3 - 2x_4 + 7x_5 &= 0 \\ 2x_1 + 5x_2 \quad\quad - x_4 + 14x_5 &= 0 \end{aligned}}$$

single matrix equation in the vector variable $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$

$$\longrightarrow \underbrace{\begin{bmatrix} 3 & 1 & -3 & 11 & 10 \\ 5 & 8 & 2 & -2 & 7 \\ 2 & 5 & 0 & -4 & 14 \end{bmatrix}}_{\text{coefficient matrix}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{0}\ \text{RHS matrix}}$$

"matrix form" of the linear system of eqns

$$x_1 \underbrace{\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}}_{\vec{v}_1} + x_2 \underbrace{\begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}}_{\vec{v}_2} + x_3 \underbrace{\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}}_{\vec{v}_3} + x_4 \underbrace{\begin{bmatrix} 11 \\ -2 \\ -1 \end{bmatrix}}_{\vec{v}_4} + x_5 \underbrace{\begin{bmatrix} 10 \\ 7 \\ 14 \end{bmatrix}}_{\vec{v}_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Q: can the zero vector $\vec{0}$ in $\mathbb{R}^3$ be expressed as a linear combination of $\vec{v}_1, \dots, \vec{v}_5$ in $\mathbb{R}^3$?

If so, this vector equation represents a linear relationship among them for each soln vector of coefficients

Note: Since there are at most 3 independent vectors in $\mathbb{R}^3$ there must be at least 2 independent relationships among 5 vectors

augmented matrix

$$\left[\begin{array}{ccccc|c} 3 & 1 & -3 & 11 & 10 & 0 \\ 5 & 8 & 2 & -2 & 7 & 0 \\ 2 & 5 & 0 & -4 & 14 & 0 \end{array}\right]$$

$\downarrow$ RREF, BackwardSubstitute, etc
(leading variables can be expressed in terms of free variables!)

The solution:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2t_1 + 3t_2 \\ t_1 - 4t_2 \\ 2t_1 + 5t_2 \\ t_1 \\ t_2 \end{bmatrix}$$

each value of (t_1, t_2) gives "a solution"

$$= t_1 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}_1} + t_2 \underbrace{\begin{bmatrix} 3 \\ -4 \\ 5 \\ 0 \\ 1 \end{bmatrix}}_{\vec{u}_2}$$

this represents the solution as a linear comb of 2 vectors $\vec{u}_1, \vec{u}_2$

The set of all possible solns is the "solution space".
It is the span of $\{u_1, u_2\}$
(all possible linear combinations of them)
Since $\{u_1, u_2\}$ is a set of 2 vectors which are linearly independent, they represent a <u>basis</u> of the solution space which is a linear subspace of $\mathbb{R}^5$
= 2-plane thru origin of $\mathbb{R}^5$

all possible linear relationships among $\vec{v}_1, \dots, \vec{v}_5$: $(-2t_1 + 3t_2)\vec{v}_1 + (t_1 - 4t_2)\vec{v}_2 + (2t_1 + 5t_2)\vec{v}_3 + t_1\vec{v}_4 + t_2\vec{v}_5 = \vec{0}$

2 independent such relationships correspond to $\vec{u}_1, \vec{u}_2$:

$$\left\{ \begin{aligned} -2v_1 + v_2 + 2v_3 + v_4 \quad\quad &= 0 \\ 3v_1 - 4v_2 + 5v_3 \quad\quad + v_5 &= 0 \end{aligned} \right.$$

(can be used to express $\vec{v}_4, \vec{v}_5$ in terms of $\vec{v}_1, \vec{v}_2, \vec{v}_3$)

(free columns can be expressed in terms of the leading columns!)

Note that setting $x_5 = -1$ leads to $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 - \vec{v}_5 = 0$
or $\underline{\vec{v}_5 = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4}$

solving this linear system for $x_1, \dots, x_4$ has same augmented matrix without final zero column.