MAT2705-01/04 05S Test 3 (Takehome) Answers

① a) $y'' + 60y' + 800y = 48 - 48e^{-10t} \rightarrow y_p = A + Be^{-10t}$

$y_p' = -10Be^{-10t}$

$y_p'' = 100Be^{-10t}$

$y_h = e^{rt} \rightarrow r^2 + 60r + 800 = 0$

$(r+20)(r+40) = 0$

$r = -20, -40 \quad e^{rt} = e^{-20t}, e^{-40t}$

$y_h = c_1 e^{-20t} + c_2 e^{-40t}$

$y_p'' + 60y_p' + 800y_p = 100Be^{-10t} + 60(-10Be^{-10t}) + 800(A + Be^{-10t})$

$= \underbrace{(100 - 600 + 800)}B e^{-10t} + \underbrace{800A}_{=48} = 48 - 48e^{-10t}$

$300B = -48$

$B = \frac{-48}{300} = -\frac{16}{100} = -0.16, \quad A = \frac{48}{800} = \frac{6}{100} = .06$

$$\boxed{y = y_h + y_p = c_1 e^{-20t} + c_2 e^{-40t} + \frac{6 - 16e^{-10t}}{100}}$$

general soln

or: $\frac{3}{50} - \frac{4}{25}e^{-10t}$

b) $y' = -20c_1 e^{-20t} - 40c_2 e^{-40t} + \frac{40}{25}e^{-10t}$

$y(0) = c_1 + c_2 + \frac{6-16}{100} = 0$

$y'(0) = -20c_1 - 40c_2 + \frac{160}{100} = -1$

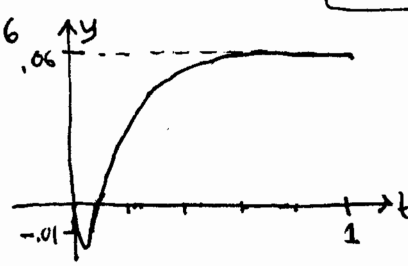
$$\begin{bmatrix} 1 & 1 \\ -20 & -40 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 10/100 \\ 60/100 \end{bmatrix}$$

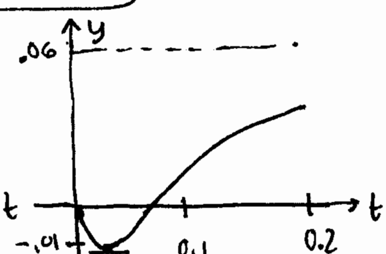
$$\begin{bmatrix} 1 & 1 & 10/100 \\ -20 & -40 & 260/100 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 7/100 \\ 0 & 1 & 3/100 \end{bmatrix} \rightarrow \begin{matrix} c_1 = 7/100 \\ c_2 = 3/100 \end{matrix}$$

$$\boxed{y = \frac{1}{100}(7e^{-20t} + 3e^{-40t} + 6 - 16e^{-10t}) = \frac{7}{100}e^{-20t} + \frac{3}{100}e^{-40t} + \frac{3}{50} - \frac{4}{25}e^{-10t}}$$

c) characteristic times are $\frac{1}{10}, \frac{1}{20}, \frac{1}{40} = \underbrace{.10}, .05, .04 \rightarrow \times 5$ char times $\sim t = .5$ so choose $t = 0..1$

$\lim_{t \to \infty} y = \frac{3}{50} = .06$





d) $y' = \frac{1}{100}\left[-140e^{-20t} - 120e^{-40t} + 160e^{-10t}\right] = 0 \rightarrow$ fsolve(%, t=0..1); $\boxed{t = 0.0271}$

e) The asymptotic value of y is 0.06 and it is pretty close to that value after 5 of the longest characteristic times.

② a) $1000X'' + 2000X' + 10{,}000X = (1000)A_0\omega^2 \sin\omega t \rightarrow \boxed{X'' + \underbrace{2}_{k_0}X' + \underbrace{10}_{\omega_0^2}X = A_0\omega^2 \sin\omega t}$

b) $\boxed{\omega_0 = \sqrt{10} \approx 3.162, \quad k_0 = 2}$ $\times \frac{\text{cycles}}{2\pi\text{ rad}} \rightarrow .503\text{ Hz}$

c) $T_0 = \frac{2\pi}{\sqrt{10}} \approx 1.987, \quad \tau_0 = 1/k_0 = 1/2, \qquad Q = \omega_0\tau_0 = \sqrt{10}/2 \approx 1.581 > 1/2$ underdamped

will find complex roots next

d) $x = e^{rt} \rightarrow r^2 + 2r + 10 = 0, \quad r = \frac{-2 \pm \sqrt{4 - 4 \cdot 10}}{2} = -1 \pm \sqrt{-9} = -1 \pm 3i$

$e^{rt} = e^{(-1 \pm 3i)t} = e^{-t}e^{\pm 3it} = e^{-t}(\cos 3t \pm i \sin 3t) \xrightarrow[\text{combos}]{\text{real}}$ Re, Im parts : $e^{-t}\cos 3t, e^{-t}\sin 3t$

$x_h = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t = \boxed{e^{-t}(c_1\cos 3t + c_2 \sin 3t)}$

e) $x'' + 2x' + 10x = \frac{9}{4}\sin 3t \rightarrow$

$10[x_p = c_3\cos 3t + c_4 \sin 3t]$

$2[x_p' = -3c_3\sin 3t + 3c_4\cos 3t]$

$1[x_p'' = -9c_3\cos 3t - 9c_4 \sin 3t]$

$x_p'' + 2x_p' + 10x_p = [(10-9)c_3 + 6c_4]\cos 3t + [-6c_3 + (10-9)c_4]\sin 3t = \frac{9}{4}\sin 3t$

$\therefore \quad c_3 + 6c_4 = 0$

$-6c_3 + c_4 = 9/4$

$$\begin{bmatrix} 1 & 6 & 0 \\ -6 & 1 & 9/4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & -27/74 \\ 0 & 1 & 9/148 \end{bmatrix} \begin{matrix} \leftarrow c_3 \\ \leftarrow c_4 \end{matrix}$$

Amp $= \frac{9}{148}\sqrt{1 + 6^2} = \frac{9\sqrt{37}}{148}$

≈ 0.370 ft

≈ 4.44 in

$$\boxed{x_p = \frac{-2 \cdot 27\cos 3t + 9\sin 3t}{148}} \approx -0.365\cos 3t + 0.061 \sin 3t$$

$x_p(0) = -2 \cdot 27/148 \quad x_h(0) = c_1$

f) $x_p' = (+2 \cdot 27 \cdot 3\sin 3t + 27\cos 3t)/148 \quad x_p'(0) = 27/148$

$x_h' = -e^{-t}(c_1\cos 3t + c_2 \sin 3t) + e^{-t}(-3c_1\sin 3t + 3c_2\cos 3t) \quad x_h'(0) = -c_1 + 3c_2$

$x(0) = c_1 - \frac{2 \cdot 27}{148} = 0$

$x'(0) = -c_1 + 3c_2 + \frac{27}{148} = 0$

augment, rref, backsub:

$c_1 = \frac{2 \cdot 27}{148}, \quad c_2 = \frac{9}{148}$

$$\boxed{x = x_h + x_p = e^{-t}(2 \cdot 27\cos 3t + 9\sin 3t)/148 + (-2 \cdot 27\cos 3t + 9\sin 3t)/148}$$

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(2) f) $x \approx \underbrace{e^{-t}(.365\cos 3t + .0608\sin 3t)}_{\text{transient soln}} + \underbrace{.0608\sin 3t - .365\cos 3t}_{\text{steady state soln}}$

g)
$$\begin{aligned} 10[\,x_p &= c_3\cos\omega t + c_4\sin\omega t\,] \\ 2[\,x_p' &= -c_3\omega\sin\omega t + c_4\omega\cos\omega t\,] \\ 1[\,x_p'' &= -\omega^2 c_3\cos\omega t - \omega^2 c_4\sin\omega t\,] \end{aligned}$$

$$x_p'' + 2x_p' + 10x_p = [(10-\omega^2)c_3 + 2\omega c_4]\cos\omega t + [-2\omega c_3 + (10-\omega^2)c_4]\sin\omega t = A_0\omega^2\sin\omega t$$

$$\begin{bmatrix} 10-\omega^2 & 2\omega \\ -2\omega & 10-\omega^2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ A_0\omega^2 \end{bmatrix}$$ augment rref backsub

$$c_1 = \frac{(-2\omega)(A_0\omega^2)}{(10-\omega^2)^2+4\omega^2}$$

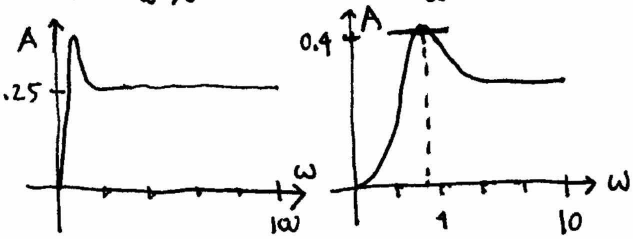
$$c_2 = \frac{(10-\omega^2)(A_0\omega^2)}{(10-\omega^2)^2+4\omega^2}$$

$$\boxed{x_p = \frac{A_0\omega^2(-2\omega\cos\omega t + (10-\omega^2)\sin\omega t)}{(10-\omega^2)^2+4\omega^2}}$$

$$A(\omega) = \sqrt{c_1^2+c_2^2} = \frac{A_0\omega^2\sqrt{4\omega^2+(10-\omega^2)^2}}{(10-\omega^2)^2+4\omega^2} = \boxed{\frac{A_0\omega^2}{\sqrt{(10-\omega^2)^2+4\omega^2}}}$$

h) Note $\lim\limits_{\omega\to\infty} A(\omega) = \lim\limits_{\omega\to\infty} \frac{A_0\omega^2}{\omega^2} = A_0 = \frac{1}{4}$ so has a horizontal asymptote.

There is a nice resonance peak around $\omega \approx 3.5$, $A \approx 0.42$



$$\frac{A'(\omega)}{A_0} = \frac{\sqrt{(10-\omega^2)^2+4\omega^2}\,(2\omega) - \omega^2\frac{1}{2}\frac{2(10-\omega^2)(-2\omega)+4(2\omega)}{\sqrt{\ldots}}}{(\sqrt{\ldots})^2}$$

$$= \frac{2\omega[\underbrace{(10-\omega^2)^2+4\omega^2+(10-\omega^2)\omega^2-2\omega^2}_{100-20\omega^2+\omega^4+4\omega^2+10\omega^2-\omega^4-2\omega^2}]}{(\ )^{3/2}}$$

$$= 100 - 8\omega^2 = 8(25/2-\omega^2)$$

$A'(\omega) = 0 \to \omega = 0$ or $\omega^2 = 25/2$, $\omega = \frac{5}{\sqrt{2}} = \boxed{\frac{5}{2}\sqrt{2} \approx 3.536}$

$$A\left(\tfrac{5}{\sqrt{2}}\right) = \frac{A_0(25/2)}{\sqrt{(10-25/2)^2+4(25/2)}} = \frac{A_0\,25/2}{\sqrt{\frac{25}{4}+50}} = \frac{A_0\,25}{\sqrt{225}} = \frac{25}{4\sqrt{225}} \approx .416667 \ (\times 12 \text{ in/ft} \to 5.0 \text{ in})$$

This is just what the plot shows. The peak frequency $3.536\,\frac{\text{rad}}{\text{sec}} = \frac{3.536}{2\pi}\,\frac{\text{cycles}}{\text{sec}} \approx 0.56$ Hz

About one oscillation every 2 sec.

(3) a) $\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}'}_{X'} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -3 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X}$ $\quad \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 27 \\ 0 \\ 0 \end{bmatrix}$

b) $\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 2 & -3-\lambda \end{vmatrix} = -(\lambda+1)(\lambda+2)(\lambda+3) = 0$

$\lambda = -1, -2, -3$

$\lambda = -1$: $A + I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 2 & -2 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccc} L & L & F \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_3 = t,\ x_1 = t,\ x_2 = t$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ E_1

$\lambda = -2$: $A + 2I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{array}{c} \begin{array}{ccc} L & L & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \end{array}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_3 = t,\ x_1 = 0,\ x_2 = \frac{1}{2}t$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$ (optional) double $E_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$\lambda = -3$: $A + 3I = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} L & L & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \ldots$ $\quad x_3 = t,\ x_1 = 0,\ x_2 = 0$ $\quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ E_3 $\quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ $\quad A_B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$

$x = By$, $y = B^{-1}x$: $\quad y' = A_B y$ $\quad \begin{aligned} y_1' &= -y_1 \\ y_2' &= -2y_2 \\ y_3' &= -3y_3 \end{aligned}$ $\quad \begin{aligned} y_1 &= c_1e^{-t} \\ y_2 &= c_2e^{-2t} \\ y_3 &= c_3e^{-3t} \end{aligned}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{-t} \\ c_2e^{-2t} \\ c_3e^{-3t} \end{bmatrix} = \begin{bmatrix} c_1e^{-t} \\ c_1e^{-t}+c_2e^{-2t} \\ c_1e^{-t}+2c_2e^{-2t}+c_3e^{-3t} \end{bmatrix}$$

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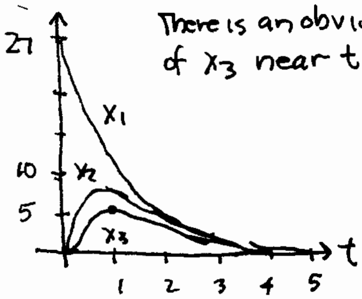
③ c) $\underline{X}(0) = \begin{bmatrix}1&0&0\\1&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}27\\0\\0\end{bmatrix}$ rref $\begin{bmatrix}1&0&0&27\\1&1&0&0\\1&2&1&0\end{bmatrix} = \begin{bmatrix}1&0&0&27\\0&1&0&-27\\0&0&1&27\end{bmatrix}$ $c_1=27$, $c_2=-27$, $c_3=27$ $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}27e^{-t}\\+27e^{-t}-27e^{-2t}\\27e^{-t}-54e^{-2t}+27e^{-3t}\end{bmatrix}$

d) $x_3 = 27(e^{-t}-2e^{-2t}+e^{-3t})$ $x_3' = 27(-e^{-t}+4e^{-2t}-3e^{-3t}) = 0$

$>$ fsolve(%, $t=0..2$); $t = 1.0986$

$x_3(1.0986) = 3.99999 \to \boxed{4.0}$

There is an obvious global max of x_3 near $t=1$



aside: this can be solved exactly →
multiply by e^{3t}:
$-e^{2t}+4e^{t}-3=0$ quadratic eq for e^t:
$(e^t)^2-4(e^t)+3=0$ $e^t = \frac{+4\pm\sqrt{16-12}}{2} = +2\pm1 = 3,1$
$t = \ln 3, \ln 1 = 0$
$x_3(\ln 3) = 27(e^{-\ln 3}-2e^{-2\ln 3}+e^{-3\ln 3})$
$= 27\left(\frac{1}{3}-\frac{2}{3^2}+\frac{1}{3^3}\right) = \frac{27}{3^3}(9-6+1) = 4$

④ a) $\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}'}_{\underline{X}'} = \underbrace{\begin{bmatrix}1&-5\\1&-1\end{bmatrix}}_{\underline{A}}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\underline{X}}$ $\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$ $\det(A-\lambda I) = \begin{vmatrix}1-\lambda&-5\\1&-1-\lambda\end{vmatrix} = (\lambda^2-1)+5 = \lambda^2+4=0$
$\lambda = \pm 2i$

$\lambda = 2i$: $\underline{A}-2i\underline{I} = \begin{bmatrix}1-2i&-5\\1&-1-2i\end{bmatrix} \to \begin{array}{cc}L&F\\ \end{array}\begin{bmatrix}1&-(1+2i)\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ $x_1=(1+2i)t$, $x_2=t$ $\begin{bmatrix}x_1\\x_2\end{bmatrix} = t\begin{bmatrix}1+2i\\1\end{bmatrix}$ $\underline{E_1}$

$\lambda=-2i$: $E_2 = \overline{E_1}$, $\underline{B} = \begin{bmatrix}1+2i&1-2i\\1&1\end{bmatrix}$ $\underline{X}=\underline{B}\underline{y}$, $\underline{y} = \underline{B}^{-1}\underline{X}$, $\underline{A_B} = \begin{bmatrix}2i&0\\0&-2i\end{bmatrix}$

$\underline{y}' = \underline{A_B}\,\underline{y}$ $\begin{bmatrix}y_1'\\y_2'\end{bmatrix} = \begin{bmatrix}2i&0\\0&-2i\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}2iy_1\\-2iy_2\end{bmatrix}$ $y_1'=2iy_1$ $y_1=c_1e^{2it}$; $y_2'=-2iy_2$ $y_2=c_2e^{-2it}$

$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1+2i&1-2i\\1&1\end{bmatrix}\begin{bmatrix}c_1e^{2it}\\c_2e^{-2it}\end{bmatrix} = c_1\underbrace{e^{2it}\begin{bmatrix}1+2i\\1\end{bmatrix}} + c_2e^{-2it}\begin{bmatrix}1-2i\\1\end{bmatrix}$

I too am careless working out the details, but I am a good debugger!

$= (\cos 2t + i\sin 2t)\begin{bmatrix}1+2i\\1\end{bmatrix} = \begin{bmatrix}\cos 2t + i\sin 2t + 2i\cos 2t - 2\sin 2t\\\cos 2t + i\sin 2t\end{bmatrix} = \begin{bmatrix}\cos 2t-2\sin 2t\\\cos 2t\end{bmatrix} + i\begin{bmatrix}\sin 2t+2\cos 2t\\\sin 2t\end{bmatrix}$

real general soln:

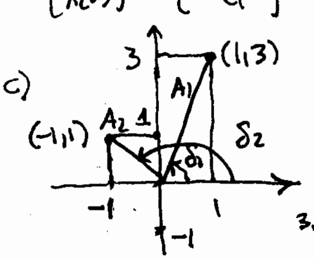
$\begin{bmatrix}x_1\\x_2\end{bmatrix} = c_1\begin{bmatrix}\cos 2t-2\sin 2t\\\cos 2t\end{bmatrix} + c_2\begin{bmatrix}\sin 2t+2\cos 2t\\\sin 2t\end{bmatrix} = \begin{bmatrix}(c_1+2c_2)\cos 2t + (-2c_1+c_2)\sin 2t\\c_1\cos 2t + c_2\sin 2t\end{bmatrix}$

b) $\omega = 2$
$T = \frac{2\pi}{\omega} = \pi$

$\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix} = \begin{bmatrix}c_1+2c_2\\c_1\end{bmatrix} = \begin{bmatrix}1\\-1\end{bmatrix}$ $c_1+2c_2=1 \to c_2 = \frac{+1+1}{2} = 1$
$c_1=-1$ $-2c_1+c_2 = 3$

$\boxed{\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}\cos 2t+3\sin 2t\\-\cos 2t+\sin 2t\end{bmatrix}}$

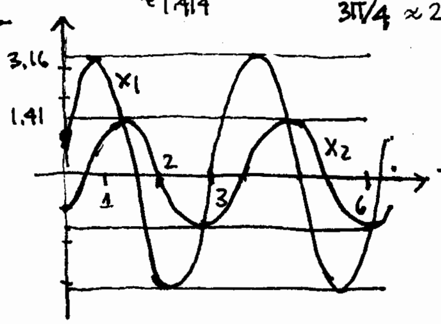
c)



arctan(1) $= \pi/4$!

$x_1 = \underset{\approx 3.162}{\sqrt{10}}\cos(2t - \overset{\approx 1.249\ \sim 71.6^\circ}{\arctan 3})$

$x_2 = \underset{\approx 1.414}{\sqrt{2}}\cos(2t - \underset{3\pi/4\ \approx 2.356\ \sim 135^\circ}{(\pi-\arctan 1)})$



d) $\frac{A_1}{A_2} = \boxed{\sqrt{5}}$ ≈ 2.236

$\delta_2-\delta_1 = \pi - \arctan 1 - \arctan 3$
$= \boxed{\frac{3\pi}{4} - \arctan 3}$ ← arctan(2) by trig identities!
$\sim \boxed{63.4^\circ}$

x_1* is ahead of x_2 by about 1/6 cycle
(x_2 lags farther behind: $\delta_2 > \delta_1$)
exactly what we expected!

* ahead in time means "earlier" in time, i.e., to the LEFT on the time line