MAT2705 TEST2 05S ANSWERS

① a)
$$\underbrace{\begin{bmatrix}0\\3\\5\end{bmatrix}}_{\underline{b}} = x_1\begin{bmatrix}1\\-1\\1\end{bmatrix}+x_2\begin{bmatrix}1\\0\\-4\end{bmatrix}+x_3\begin{bmatrix}0\\2\\0\end{bmatrix} = \underbrace{\begin{bmatrix}1&1&0\\-1&0&2\\1&-4&0\end{bmatrix}}_{\underline{A}}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\underline{x}}$$

b)
$$\text{augment}(\underline{A},\underline{b}) = \begin{bmatrix}1&1&0&0\\-1&0&2&3\\1&-4&0&5\end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix}1&0&0&1\\0&1&0&-1\\0&0&1&2\end{bmatrix}$$

$\to x_1=1,\ x_2=-1,\ x_3=2$

so $\boxed{[0,3,5] = [1,-1,1]-[1,0,-4]+2[0,2,0]}$

$\left[\begin{aligned}\text{check: } &= [1-1+0,\ -1-0+4,\ 1-(-4)+0]\\ &= [0,3,5]\ ✓\end{aligned}\right]$

b) Since their augmented matrix $\underline{A}$ row reduces to the identity, they are linearly independent and hence form a basis of $\mathbb{R}^3$, so yes, any vector in $\mathbb{R}^3$ can be expressed uniquely as a linear combination of these 3 vectors.

alternatively $\det(\underline{A}) = 10 \neq 0$ implies the same conclusion.

[in fact $\underline{A}$ has an inverse so $\underline{A}\,\underline{y}=\underline{x}$ has the soln $\underline{y}=\underline{A}^{-1}\underline{x}$ for the necessary coeff's]

② a) $(D^3+D^2+4D+4)y=0 \quad \leftarrow y=e^{rx}$

$(r^3+r^2+4r+4)y=0$

$(r^3+r^2+4r+4)=0$

$(r+1)(r^2+4)=0 \to r=-1,\ \pm 2i$

$y = e^{-x},\ e^{\pm 2i} = \cos 2x \pm i\sin 2x$

real basis: $e^{-x},\ \cos 2x,\ \sin 2x$

gen soln: $\boxed{y = c_1e^{-x}+c_2\cos 2x+c_3\sin 2x}$

b) $y' = -c_1e^{-x}-2c_2\sin 2x+2c_3\cos 2x$

$y'' = c_1e^{-x}-4c_2\cos 2x-4c_3\sin 2x$

$$\begin{aligned} y(0) &= c_1+c_2 &&= 0\\ y'(0) &= -c_1 + 2c_3 &&= 3\\ y''(0) &= c_1-4c_2 &&= 5\end{aligned} \to \begin{bmatrix}1&1&0\\-1&0&2\\1&-4&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\3\\5\end{bmatrix}$$

same as problem 1.

c) $c_1=1,\ c_2=-1,\ c_3=2$

4· $\boxed{y = e^{-x}-\cos 2x+2\sin 2x}$

4· $y' = -e^{-x}+2\sin 2x-4\cos 2x$

1· $y'' = e^{-x}+4\cos 2x+8\sin 2x$

1· $y''' = -e^{-x}-8\sin 2x+16\cos 2x$

$\hookrightarrow y'''+y''+4y'+4y = \underbrace{(4-4+1-1)}_{0}e^{-x}+\underbrace{(-4+8+4-8)}_{0}\cos 2x+\underbrace{(8-16+8+16)}_{0}\sin 2x = 0$ ✓

③ b)
$$\begin{aligned} n_1 - n_3 - 2n_4 &= 0\\ 4n_1 - 2n_4-2n_5 &= 0\\ 2n_2-2n_3 - n_5 &= 0\end{aligned}$$

$$\begin{bmatrix}1&0&-1&-2&0\\4&0&0&-2&-2\\0&2&-2&0&-1\end{bmatrix}\begin{bmatrix}n_1\\n_2\\n_3\\n_4\\n_5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{pmatrix}1&0&-1&2&0&0\\4&0&0&-2&-2&0\\0&2&-2&0&1&0\end{pmatrix} \xrightarrow{\text{rref}} \begin{array}{cccccc} L&L&L&F&F&\\ n_1&n_2&n_3&n_4&n_5& \end{array}\begin{bmatrix}1&0&0&-1/2&-1/2&0\\0&1&0&3/2&-1&0\\0&0&1&3/2&-1/2&0\end{bmatrix}$$

$$\left.\begin{aligned} n_1-\tfrac12 n_4-\tfrac12 n_5&=0\\ n_2+\tfrac32 n_4 - n_5&=0\\ n_3+\tfrac32 n_4-\tfrac12 n_5&=0\\ n_4 &= t_1\\ n_5&=t_2\end{aligned}\right\} \xrightarrow{\text{backsub}} \begin{aligned} n_1 &= \tfrac12 t_1+\tfrac12 t_2\\ n_2&=-\tfrac32 t_1+t_2\\ n_3&=-\tfrac32 t_1+\tfrac12 t_2\end{aligned}$$

$$\boxed{[n_1,n_2,n_3,n_4,n_5] = [\tfrac12 t_1+\tfrac12 t_2,\ -\tfrac32 t_1+t_2,\ -\tfrac32 t_1+\tfrac12 t_2,\ t_1,\ t_2]}$$

$$= t_1\underbrace{[\tfrac12,-\tfrac32,-\tfrac32,1,0]}_{\vec{v}_1}+t_2\underbrace{[\tfrac12,1,\tfrac12,0,1]}_{\vec{v}_2}$$

$\hookrightarrow$ double:

$\vec{u}_1 = [1,-3,-3,2,0]$

$\vec{u}_2 = [1,2,1,0,2] \leftarrow$ simple reaction

a) $[n_1,n_2,n_3,n_4,n_5] = [4,3,0,2,6]$

$n_1-n_3-2n_4 = 4-0-2\cdot 2 = 4-4=0$ ✓

$4n_1-2n_4-2n_5 = 4(4)-2(2)-2(6) = 16-4-12=0$ ✓

$2n_2-2n_3-n_5 = 2(3)-2(0)-6 = 6-6=0$ ✓

c)
$$x_1\begin{bmatrix}1\\-3\\-3\\2\\0\end{bmatrix}+x_2\begin{bmatrix}1\\2\\1\\0\\2\end{bmatrix} = \begin{bmatrix}1&1\\-3&2\\-3&1\\2&0\\0&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}4\\3\\0\\2\\6\end{bmatrix}$$

$$\begin{pmatrix}1&1&4\\-3&2&3\\-3&1&0\\2&0&2\\0&2&6\end{pmatrix}\xrightarrow{\text{rref}} \begin{array}{cc} x_1 & x_2 \end{array}\begin{pmatrix}1&0&1\\0&1&3\\0&0&0\\0&0&0\\0&0&0\end{pmatrix} \to \begin{aligned}x_1&=1\\x_2&=3\end{aligned}$$

so $\boxed{\vec{u}_3 = \vec{u}_1+3\vec{u}_2}$