MAT1505-05 04S Final Exam Answers

① a) $\int_{x=0}^{x=\ln 3} e^x (1+e^x)^{1/2}\, dx$ $\quad$ $e^x dx = du$

$u = 1+e^x$ $\quad$ $x=\ln 3 \to u = 1+e^{\ln 3} = 1+3 = 4$; $\quad x=0 \to u = 1+e^0 = 1+1 = 2$

$\frac{du}{dx} = e^x, \; du = e^x dx$

$= \boxed{\int_2^4 u^{1/2}\, du}$

b) using Maple: > int(..., x=a..b); gives the result $\boxed{\frac{16-4\sqrt{2}}{3} \approx 3.447715251}$

c) $= \frac{2}{3}u^{3/2}\Big|_2^4 = \frac{2}{3}\left(4^{3/2} - 2^{3/2}\right) = \frac{2}{3}\left(2^3 - 2\cdot 2^{1/2}\right)$

$= \boxed{\frac{16-4\sqrt{2}}{3} \checkmark \approx 3.447715}$. yes, they agree.

② a) $\int \underbrace{x}_{u}\, \underbrace{\sin\frac{x}{2}\, dx}_{dv} = \underbrace{x}_{u}\underbrace{(-2\cos\frac{x}{2})}_{v} - \int \underbrace{-2\cos\frac{x}{2}}_{v}\, \underbrace{dx}_{du}$

$u = x, \quad dv = \sin\frac{x}{2}\, dx$

$du = dx, \quad v = -2\cos\frac{x}{2}$

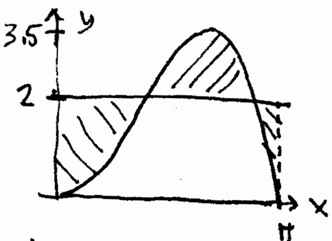
$+ 2\cdot 2\sin\frac{x}{2} + C$

$= \boxed{-2x\cos\frac{x}{2} + 4\sin\frac{x}{2} + C}$

b) $f_{avg} = \frac{1}{2\pi}\int_0^{2\pi} x\sin\frac{x}{2}\, dx = \frac{1}{2\pi}\left(-2x\cos\frac{x}{2} + 4\sin\frac{x}{2}\right)\Big|_0^{2\pi}$

$= \frac{1}{2\pi}\left[-4\pi \underbrace{\cos\pi}_{-1} + 4\underbrace{\sin\pi}_{0} + 0 - 4\sin 0\right] = \boxed{2}$

c) first zero of sin is at π! $\quad \frac{x}{2} = \pi \to x = 2\pi$



The area below $y=2$ seems about equal to the area above $y=2$, so the rectangle has about the same area as under the function.

d) $\frac{dy}{dx} = y^2 x \sin\frac{x}{2}$ $\quad$ separable D.E.

$\int y^{-2}\, dy = \int x\sin\frac{x}{2}\, dx$ sep and integrate

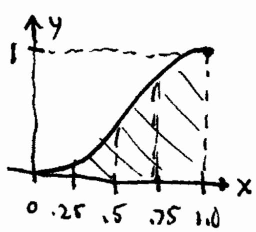
$-y^{-1} = 4\sin\frac{x}{2} - 2x\cos\frac{x}{2} + C$

$y = \frac{-1}{4\sin\frac{x}{2} - 2x\cos\frac{x}{2} + C}$

$x=0 \to y=1: \quad 1 = \frac{-1}{0+0+C} = -\frac{1}{C} \to C = -1$

$\boxed{y = \frac{-1}{4\sin\frac{x}{2} - 2x\cos\frac{x}{2} - 1} = \frac{1}{1 - 4\sin\frac{x}{2} + 2x\cos\frac{x}{2}}}$

③ b)



area looks like half the unit square ~ 0.5

③ a) $\Delta x = \frac{1-0}{4} = \frac{1}{4} = .25, \quad f(x) = \sin\frac{\pi x^2}{2}$

$S_4 = \frac{1}{12}\left(\underbrace{f(0)}_{0} + 4\underbrace{f(.25)}_{\approx .0980171434} + 2\underbrace{f(.5)}_{\approx .3826834325} + 4\underbrace{f(.75)}_{\approx .7730104534} + \underbrace{f(1)}_{\approx 1}\right)$

$\approx \boxed{0.437456}$

c) By Maple, $I = \text{FresnelS}(1)$ $\approx \boxed{0.4382591474}$

$I - S_4 \approx 0.000803 < \frac{1}{2}\times 10^{-2}$

$\not< \frac{1}{2}\times 10^{-3}$

so expect only $\boxed{\text{2 decimal place accuracy}}$.

indeed the third digit is wrong.

④ a) $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)(x-1)^n}{n!}$

b) $f(x) = x + x^{-1}$ $\quad$ $f(1) = 2$

$f'(x) = 1 - x^{-2}$ $\quad$ $f'(1) = 0$

$f''(x) = (-1)(-2)x^{-3}$ $\quad$ $f''(1) = 2!$

$f'''(x) = (-1)(-2)(-3)x^{-4}$ $\quad$ $f^{(3)}(1) = -3!$

$f^{(4)}(x) = (-1)(-2)(-3)(-4)x^{-5}$ $\quad$ $f^{(4)}(1) = 4!$

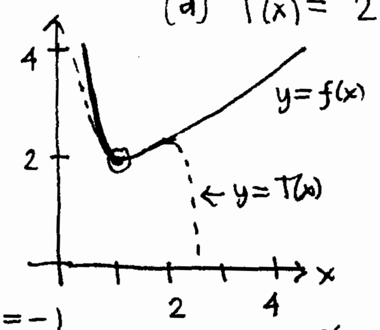
$f^{(5)}(x) = (-1)(-2)(-3)(-4)(-5)x^{-6}$ $\quad$ $f^{(5)}(1) = -5!$

$f(x) = 2 + 0(x-1) + \frac{2!(x-1)^2}{2!} - \frac{3!(x-1)^3}{3!} + \frac{4!(x-1)^4}{4!} - \frac{5!(x-1)^5}{5!}$

$= \boxed{2 + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5 + \ldots}$

(c) $= \boxed{2 + \sum_{n=2}^{\infty}(-1)^n (x-1)^n}$

(d) $T(x) = 2 + (x-1)^2 - (x-1)^3 + (x-1)^4 - (x-1)^5$



At $x=1$ tan line horizontal, concave up, fits value $f(1)=2$ and 2 curves coincide to pixel scale on interval $.5<x<1.5$ so this looks good!

⑤ a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$

b) $e^x - 1 - xe^x = $

$(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots) - 1 = -\frac{1}{2}x^2 - \frac{1}{3}x^3 + \ldots$

$-(x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4 + \ldots)$

$(e^x - 1)^2 = (x + \frac{1}{2}x^2 + \ldots)^2 = x^2 + x^3 + \frac{1}{4}x^4 + \ldots$

$\frac{e^x - 1 - xe^x}{(e^x-1)^2} = \frac{-\frac{1}{2}x^2 - \frac{1}{3}x^3 + \ldots}{x^2 + x^3 + \ldots}$

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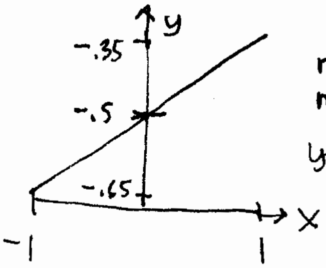
5 b) $\lim_{x\to 0} \frac{e^x - 1 - xe^x}{(e^x-1)^2} = \lim_{x\to 0} \frac{-\frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots /x^2}{x^2 + x^3 + \dots \ /x^2} = \lim_{x\to 0} \frac{-\frac{1}{2} - \frac{1}{3}x + \dots}{1 + x + \dots} = \frac{-1/2}{1} = \boxed{-\frac{1}{2}}$

c)



right on the money!

yes, consistent.

6 a) $\int \frac{x}{4+x^2}\,dx = \int \frac{du/2}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(4+x^2) + C.$

$u = 4+x^2$
$\frac{du}{dx} = 2x$
$du = 2x\,dx$
$\frac{du}{2} = x\,dx$

b) $\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{|x|^{2(n+1)}}{2(n+1)4^{n+1}}}{\frac{|x|^{2n}}{2n\,4^n}} = \frac{|x|^{2n+2}}{|x|^{2n}} \frac{4^n}{4^{n+1}} \frac{2n}{2(n+1)} = \frac{|x|^2}{4}\left(\frac{n}{n+1}\right)$

$\xrightarrow{n\to\infty} \frac{|x|^2}{4} < 1$ for convergence

$|x|^2 < 4 \to |x| < \boxed{2 = R} \to -2 \overset{?}{\le} x \overset{?}{\le} 2$ interval of convergence?

$x = 2$: $\sum_{n=1}^{\infty} (-1)^{n-1} \underbrace{\frac{2^{2n}}{2n4^n}}_{=1} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} (-1)^{n-1}\frac{1}{n}$

alternating harmonic series converges since $\frac{1}{n}$ decreases to 0 as $n\to\infty$ (ALT SERIES TEST).

$x = -2$ $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-2)^{2n}}{2n\cdot 4^n} = \sum (-1)^{n-1} \frac{2^{2n}}{2n\,4^n}$ same as before. since $(-1)^{2n} = ((-1)^2)^n = 1^n = 1$ even powers of -1 are 1. converges

so interval of convergence: $-2 \le x \le 2.$

c) $0 \le x \le 1$ is inside the interval of convergence so the series representation of the function and the integration of it remain valid.

d) $\int_0^1 \frac{x}{4+x^2}\,dx = \frac{1}{2}\ln(4+x^2)\Big|_0^1 = \boxed{\frac{1}{2}(\ln 5 - \ln 4) \approx 0.111572}$

e) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n\cdot 4^n}\Big|_0^1 = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2n\cdot 4^n} = \frac{1}{2\cdot 4} - \frac{1}{(2\cdot 2)4^2} + \frac{1}{2\cdot 3\cdot 4^3} - \frac{1}{2\cdot 4\cdot 4^4} + \frac{1}{2\cdot 5 \cdot 4^5} - \dots$

$= \underbrace{0.1250000 - 0.0156250 + 0.0026042}_{} - \underbrace{0.0004883}_{} + 0.0000977 - \dots$

S_3 sufficient $\approx \boxed{0.111979}$ $|a_4| < \frac{1}{2}\times 10^{-3}$

first 3 terms minimum necessary for 3 dec place accuracy (~ 0.112)

f) 0.111572 "exact"
-0.111979 "approx"
-0.000407 — abs value of error is less than $\frac{1}{2}\times 10^{-3}$ as expected.