MAT1505-05 04S Final Exam Print Name (Last, First) ______________________________|___

Show all work, including mental steps, in a clearly organized way that speaks for itself. Use proper mathematical notation, identifying expressions by their proper symbols (introducing them if necessary), and use arrows and equal signs when appropriate. Always simplify expressions. BOX final short answers. LABEL parts of problem. Keep answers exact (no decimal approximations, if possible). [See long instructions on reverse].

① $\int_0^{\ln 3} e^x (1+e^x)^{1/2}\,dx$ a) Using an appropriate u-substitution convert this to a completely new integral expressed entirely in the new integration variable u without actually finding an antiderivative of the new integrand.
b) Use technology to evaluate both the original integral and your new integral either exactly or numerically to make sure they agree. Give the values you find and state how you found them.
c) Now finish evaluating your new integral exactly by hand and then give its decimal approximation to 6 decimal places. Does your number agree with part b) ?

② a) Evaluate $\int x \sin(x/2)\,dx$ by hand.
b) Use your result to evaluate the average value of the function $f(x) = x \sin\frac{x}{2}$ on the interval from 0 to $a>0$, where $a$ is the first positive zero of the function (i.e. $f(a)=0$).
c) Make a rough sketch (label the tickmarks!) of $f$ and its average value $f_{avg}$ (as a constant function) on this interval from a technology plot. Does $f_{avg}$ look right? Explain.
d) Use your result from part a) to solve the initial value problem:
$\frac{dy}{dx} = y^2 x \sin\frac{x}{2}$, $y(0)=1$. State your final answer in the form $y=y(x)$.

③ $I = \int_0^1 \sin\frac{\pi x^2}{2}\,dx$ a) Recalling $S_4 = \frac{\Delta x}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + f_4)$, use Simpson's rule with $n=4$ divisions to approximate the integral $I$, giving your result to 6 decimal places. State all values that go into evaluating the final result.
b) Make a hand sketch of an appropriate technology plot illustrating the function and division of the interval (give tickmark values!).
c) Evaluate the error $I - S_4$ by using a technology value for $I$. How many decimal places are accurate in your Simpon approximation?

④ a) State the Taylor series formula for a function $f(x)$ expressed as a Taylor series centered at $x=1$.
b) Use this formula to find the first 5 nonzero terms in the Taylor series of $f(x) = x + x^{-1}$ centered at $x=1$. Give the simplified rational number coefficients in your final expression.
c) By observing what happens in evaluating these first 5 terms (don't multiply out factors!) express the complete Taylor series.
d) Let $T(x)$ be the truncated polynomial consisting of these 5 terms. Plot it together with the original function in an appropriate window centered on $x=1$. Make a rough handsketch of what you see, labeling the two curves (tickmarks too!).
Explain whether they seem consistent. (value, slope, concavity sign at $x=1$?)

OPTIONAL

⑤ a) Write down the Taylor series for $e^x$ centered at $x=0$ (or rederive it if you forget) in sigma notation and then give its first 4 terms explicitly.

b) The Bernoulli number $B_1$ can be defined by $B_1 = \lim_{x\to 0} \dfrac{e^x - 1 - xe^x}{(e^x-1)^2} = \lim_{x\to 0} \dfrac{f(x)}{g(x)}$.

Evaluate the first two nonzero terms of the Taylor series (at $x=0$) for $f(x)$ and $g(x)$: $f(x) = f_2(x) + \cdots$, $g(x) = g_2(x) + \cdots$ and use this representation of $f(x)$ and $g(x)$ to evaluate the limit exactly (no decimal equivalent!). Make it clear how you evaluate this limit.

c) Is your result consistent with a technology plot of $f(x)/g(x)$? Handsketch your supporting plot and explain.

⑥ By using the geometric series formula we can establish the identity

$$\int \frac{x}{4+x^2}\,dx = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n\cdot 4^n} + C$$

a) Evaluate the lefthand side by hand integration. (Check by differentiation that it is correct.)

b) Find the radius $R$ of convergence and the complete interval of convergence of the right hand side (don't forget to check endpoint convergence and justify your conclusions).

Consider next the definite integral: $\displaystyle\int_0^1 \frac{x}{4+x^2}\,dx = \left[\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{2n\cdot 4^n}\right]\Bigg|_0^1$

c) Given your answer to part b), is this valid to do for $0 \le x \le 1$? Explain.

d) Evaluate the left hand side exactly using part a) and then give its decimal approximation to 6 decimal places.

e) Use term by term integration ⊗ to evaluate the righthandside to an explicit infinite series. Then evaluate (and state their values) all terms necessary to establish the lowest partial sum which approximates the series sum to 3 decimal places. How many terms must be summed? What is their sum to 6 decimal places?

f) Using your result for the left hand side, compute the actual error in this approximation of the right hand side. Is this error consistent with your estimate of the error used in part e)? Explain.

⊗ the integration step is already done — only limit of integration evaluation remains!

LONG INSTRUCTIONS ARE ON THE CLASS WEB SITE:

http://www.homepage.villanova.edu/robert.jantzen/mat1505/