Mat1500-04/06 04F Test 1 Answers

① a) $\lim_{t\to\infty} P = \lim_{t\to\infty} \frac{10000}{1+19e^{-t/5}} \underset{\uparrow}{=} \frac{10000}{1+19\underbrace{\lim_{t\to\infty} e^{-t/5}}} = \frac{10000}{1+19(0)} = \boxed{10,000}$

no problem rule — decaying exponential goes to zero

b) $P = \frac{10,000}{1+19e^{-t/5}}$

$\frac{P}{10,000} = \frac{1}{1+19e^{-t/5}}$

$\frac{10,000}{P} = 1+19e^{-t/5}$

$\frac{10,000}{P} - 1 = 19e^{-t/5}$

$\frac{\left(\frac{10000}{P}-1\right)}{19} = e^{-t/5}$

$\ln\left(\frac{\frac{10000}{P}-1}{19}\right) = -\frac{t}{5}$

$\boxed{t = -5\ln\left(\frac{\left(\frac{10000}{P}-1\right)}{19}\right) = -5\ln\left(\frac{10000-P}{19P}\right)} = f^{-1}(P)$

c) $P = 1000 \to t = f^{-1}(1000)$

$= -5\ln\left(\frac{10-1}{19}\right) = \boxed{-5\ln\left(\frac{9}{19}\right)}$ exact

$\approx \boxed{3.74 \text{ months}}$ (at most 2 significant digits in problem so only keep 3)

② Division by 0 occurs at $x=25$ in the first expression so its limit there must be checked to see if it agrees with the function value at $x=25$, otherwise this is continuous everywhere else: $\lim_{x\to 25} f(x)$

$= \lim_{x\to 25} \frac{\sqrt{x}-5}{x-25}\frac{(\sqrt{x}+5)}{(\sqrt{x}+5)} = \lim_{x\to 25}\frac{(x-25)}{(x-25)(\sqrt{x}+5)} = \lim_{x\to 25}\frac{1}{\sqrt{x}+5} = \frac{1}{\sqrt{25}+5} = \frac{1}{5+5} = \frac{1}{10}$

but $f(25) = \frac{1}{5}$ so $\lim_{x\to 25} f(x) \neq f(25)$ so f is not continuous there (it has a hole where it has the "wrong value"). Finally $x \geq 0$ for sqrt to be real, but f is continuous from the right there (no problem).

③ $y = 2x - x^2 = f(x)$

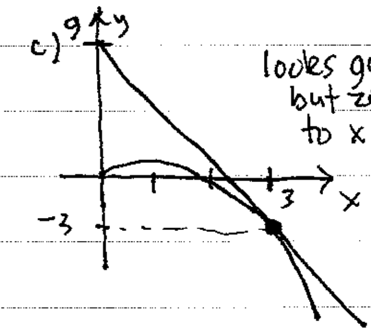
a) $m_{tan} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{2(a+h)-(a+h)^2-[2a-a^2]}{h} = \lim_{h\to 0}\frac{\cancel{2a}+2h-\cancel{a^2}-2ah-h^2-\cancel{2a}+\cancel{a^2}}{h}$

$= \lim_{h\to 0}\frac{2h-2ah-h^2}{h} = \lim_{h\to 0}\frac{\cancel{h}(2-2a-h)}{\cancel{h}} = \lim_{h\to 0} 2-2a-h = 2-2a-0 = \boxed{2-2a}$

b) $x=3 \to m_{tan} = 2-2(3) = -4$
$y = f(3) = 2(3)-3^2 = -3$ pt $(3,-3)$

$y-(-3) = -4(x-3)$
$y+3 = -4(x-3)$ $\to \boxed{y = -3-4(x-3)}$ or
$= -3-4x+12$
$\boxed{y = 9-4x}$

c)



looks good to me but zooming in to $x = 2.99..3.01$ is very convincing

④ a) Division by 0 occurs at $x=2$. No other problems occur so the domain is $\boxed{x \neq 2 \text{ or } (-\infty,2)\cup(2,\infty)}$.

b) $\lim_{x\to\pm\infty}\frac{2x^2-6x+4}{x^3-2x^2+4x-8}\frac{/x^3}{/x^3} = \lim_{x\to\pm\infty}\frac{2/x - 6/x^2 + 4/x^3}{1-2/x+4/x^2-8/x^3}$

$= \frac{0}{1} = \boxed{0}$ since negative powers of x go to zero.

so $\boxed{y=0}$ is an asymptote in both directions.

c) $\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{2(x-1)}{(x-2)^2}$ (numerator $\to 2$, denominator $\to 0$)

so must go infinite, only need to control sign. numerator is positive near 2 and denominator is always positive so result must be $\boxed{\infty}$. so $\boxed{x=2}$ is the vertical asymptote.

⑤ $f(0) = e^{-0^2} - 0 = e^0 = 1 > 0$
$f(1) = e^{-1}-1 \approx -0.632 < 0$

This function, as a composition and difference of everywhere continuous functions, is continuous on the interval and so must assume every value between 1 and -0.632 somewhere in the interior. In particular the intermediate value 0 must be assumed at some x between 0 and 1, so the equation must have a solution there.

TEST 1 FINAL COMMENT

The final point I wanted to make on asymptotes but for which no time remained was the following:

Suppose instead of a 1 in the factored formula for problem 4 we had a parameter a:

$$f(x) = \frac{2x^2 - 2(a+2)x + 4a}{x^3 - 6x^2 + 12x - 8} = \frac{2(x-a)(x-2)}{(x-2)^3} = \frac{2(x-a)}{(x-2)^2}$$

For f to have a vertical asymptote at $x=2$, it is enough that the numerator not be zero so first assume $a \neq 2$:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{2(x-a)}{(x-2)^2} \begin{cases} = +\infty \\ = -\infty \end{cases}$$

$2(x-a) \to 2(2-a)$ ← sign depends on a: If $a<2$, then positive, so the whole quotient is positive; if $a>2$, then negative, so the whole quotient is negative.

$(x-2)^2 \to 0$ but always positive because of square (so $\lim_{x\to 2^-}$ same)

$$\lim_{x\to 2^-} f(x) \;/\!/$$

Here we can no longer graph the formula with technology because we don't have a definite value of a, but we can do the simple <u>reasoning</u> that EXPLAINS what we see when $a=1$.

Even if we tried plotting this for a set of values of a, we would need to think wisely about which values to plot and what we could conclude from those special values. It is simpler to reason directly with the signs of the factors.

Finally suppose $a=2$:

$$f(x) = \frac{2(x-2)(x-2)}{(x-2)^3} = \frac{2}{x-2} \quad \begin{matrix} \xrightarrow{x\to 2^+} +\infty \\ \xrightarrow{x\to 2^-} -\infty \end{matrix}$$

now the sign of $x-2$ determines which ∞ we approach on either side, so we get a completely different behavior at the vertical asymptote.

We are trying to develop some elementary math reasoning skills so we can <u>explain</u> what we see in technology output. If we can do this for concrete functions with no parameters, we can then do it for functions involving parameters where numerics & graphics do not immediately apply. We have therefore broadened our analytical skills.

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③ a) factor method (less straightforward) $\quad f(x) = 2x - x^2$ $\quad (x-a)(x+a)$

$$m_{tan} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{2x-x^2-(2a-a^2)}{x-a} = \lim_{x\to a} \frac{2(x-a)-(x^2-a^2)}{(x-a)}$$

$$= \lim_{x\to a} \frac{(x-a)[2-(x+a)]}{x-a} = \lim_{x\to a} 2-x-a = \boxed{2-2a}$$