mat1500-03/10 02F finalexamanswers (1)

① $y = (3x-2)^4 (2x-3)^3$

$\frac{dy}{dx} = \frac{d}{dx}(3x-2)^4 \cdot (2x-3)^3 + (3x-2)^4 \frac{d}{dx}(2x-3)^3$

$= 4(3x-2)^3 \underbrace{\frac{d}{dx}(3x-2)}_{3} \cdot (2x-3)^3 + (3x-2)^4\, 3(2x-3)^2 \underbrace{\frac{d}{dx}(2x-3)}_{2}$

$= 12(3x-2)^3(2x-3)^3 + 6(3x-2)^4(2x-3)^2$

$= 6(3x-2)^3(2x-3)^2 [2(2x-3) + (3x-2)]$

$4x-6+3x-2 = 7x-8$

$= 6(3x-2)^3(2x-3)^2(7x-8) = 0$

$3x-2=0$ $2x-3=0$ $7x-8=0$

$x=\frac{2}{3}$ $x=\frac{3}{2}$ $x=\frac{8}{7}$

critical numbers x: $\boxed{\frac{2}{3}, \frac{3}{2}, \frac{8}{7}}$

can also be done by logarithmic differentiation

② $s'(t) = 2 \cdot 2\sin t \underbrace{\frac{d}{dt}\sin t}_{\cos t} + \sin(4t) \underbrace{\frac{d}{dt}(4t)}_{4}$

$= 4\sin t\cos t + 4\sin(4t)$

$s''(t) = 4 \underbrace{\frac{d}{dt}(\sin t)}_{\cos t} \cos t + 4\sin t \underbrace{\frac{d}{dt}\cos t}_{-\sin t} + 4\cos(4t)\underbrace{\frac{d}{dt}(4t)}_{4}$

$= 4(\cos^2 t - \sin^2 t) + 16\cos(4t)$

$s''(\frac{\pi}{2}) = 4(\underbrace{\cos^2\frac{\pi}{2}}_{0} - \underbrace{\sin^2\frac{\pi}{2}}_{1}) + 16\underbrace{\cos(\overbrace{4\cdot\frac{\pi}{2}}^{2\pi})}_{1}$

$= -4+16 = \boxed{12}$

③ $\frac{d}{dx}[x^{1/2} + y^{1/2} = 3]$ $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x^{-1/2}/2}{y^{-1/2}/2} = -\frac{y^{1/2}}{x^{1/2}}$ $\left.\frac{dy}{dx}\right|_{(4,1)} = -\frac{1^{1/2}}{4^{1/2}} = -\frac{1}{2}$

$y-1 = -\frac{1}{2}(x-4) \rightarrow y = 1-\frac{1}{2}(x-4) = 1-\frac{1}{2}x+2$

$\boxed{y = 3-\frac{1}{2}x}$

L'Hopital's rule justified

④ a) $\lim_{x\to 0} \frac{\sin x \to 0}{x \to 0} = \lim_{x\to 0} \frac{\frac{d}{dx}\sin x}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = \boxed{1}$

b) $f(x) = \frac{\sin x}{x}$ $f'(x) = \frac{x\overbrace{\frac{d}{dx}\sin x}^{\cos x} - \sin x \overbrace{\frac{d}{dx}x}^{1}}{x^2} = \frac{x\cos x - \sin x}{x^2}$

LH justified

$\lim_{x\to 0}\frac{x\cos x - \sin x \to 0}{x^2 \to 0} = \lim_{x\to 0}\frac{\frac{d}{dx}(x\cos x - \sin x)}{\frac{d}{dx}(x^2)}$

$= \lim_{x\to 0}\frac{1\cos x - x\sin x - \cos x}{2x} = \lim_{x\to 0}\frac{-x\sin x}{2x} = -\lim_{x\to 0}\frac{+\sin x}{2} =$

$= \frac{-\sin 0}{2} = \boxed{0}$

⑤ a) $f(x) = x - x^{1/2}$ $f'(x) = 1-\frac{1}{2}x^{-1/2} = 0$

$\rightarrow 2 = x^{-1/2} \rightarrow x = (\frac{1}{2})^2 = \frac{1}{4}$ critical value in (0,4)

$f''(x) = 0 - \frac{1}{2}(-\frac{1}{2})x^{-3/2} = \frac{1}{4}x^{-3/2} > 0$ ☺

local min by 2nd derivative test

$f(\frac{1}{4}) = \frac{1}{4} - (\frac{1}{4})^{1/2} = \frac{1}{4}-\frac{1}{2} = \boxed{-\frac{1}{4} \text{ local min value}}$

$f(0) = 0-0^{1/2} = 0$

$f(4) = 4-4^{1/2} = 4-2 = 2$

global max/min can only occur at endpoints or local max/min → $\boxed{\text{global max value is } 2}$ (at $x=4$)

$\boxed{\text{global min value is } -1/4}$ (at $x=\frac{1}{4}$)

b) $f(4) = 2$ $f'(4) = 1-\frac{1}{2\sqrt{4}} = 1-\frac{1}{4} = \frac{3}{4}$

$y-2 = \frac{3}{4}(x-4) \rightarrow y = 2+\frac{3}{4}(x-4)$

$\boxed{L(x) = 2+\frac{3}{4}(x-4) = \frac{3x}{4}-1}$

$f(3.9) \approx L(3.9) = 2 + \underbrace{\frac{3}{4}}_{.75}\underbrace{(3.9-4)}_{-0.1}$

$= 2 - .075 = \boxed{1.925}$

c) $f''(4) = \frac{1}{4}4^{-3/2} = \frac{1}{4\cdot 2^3} = \frac{1}{32} > 0$

f curves up away from tangent line, so linear approximation is $\boxed{\text{too low.}}$

⑥ $V = (L+L^{-1})^{1/2}$ $1-L^{-2}$

$\frac{dV}{dL} = \frac{1}{2}(L+L^{-1})^{-1/2}\frac{d}{dL}(L+L^{-1})$

$\frac{dV}{dL} = \frac{1-L^{-2}}{2(L+L^{-1})^{1/2}}$ $dV = \frac{1-L^{-2}}{2(L+L^{-1})^{1/2}}\, dL$

$\frac{dV}{V} = \frac{1-L^{-2}}{2(L+L^{-1})}\, dL$

$L=2$, $dL = (.10)(2) = .2$ (inc by 10%)

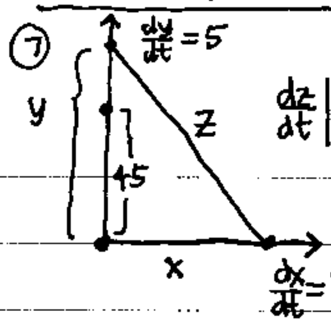
$\frac{dV}{V} = \frac{1-1/4}{2(2+\frac{1}{2})}(.2) = \frac{3/4}{2(5/2)}(0.2) = \frac{3}{20}(0.2)$

$= \frac{3}{10}(0.1) = \frac{3}{100} = .03 \rightarrow \boxed{3\%}$

(increase since positive)

mat1500-0310 02F final exam answers (2)

⑦



$\left.\frac{dz}{dt}\right|_{t=3} = ?$ $\frac{d}{dt}\left[z^2 = x^2+y^2\right]$

$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$\frac{dz}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{z}$

$t=3$: $x = 15\cdot 3 = 45$ $\quad y = 45 + 3\cdot 5 = 60$

$z^2 = 45^2 + 60^2 = (15\cdot 3)^2 + (5\cdot 4)^2$

$= 15^2(3^2+4^2) = 15^2\cdot 5^2$, $z = 5\cdot 15 = 75$

ft

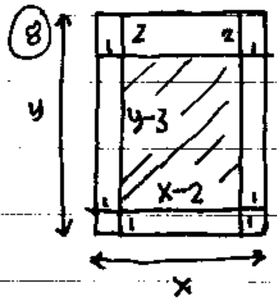
$\left.\frac{dz}{dt}\right|_{t=3} = \frac{45\cdot 15 + 60\cdot 5}{75} = \frac{\cancel{15}(45+4\cdot 5)}{\cancel{15}\cdot 5}$

sec

$= 9 + 4 = 13$

$\boxed{\text{distance is increasing at } 13 \text{ ft/s}}$

⑧



$xy = 180 \to y = \frac{180}{x}$

$A = (x-2)(y-3)$ max

$A(x) = (x-2)\left(\frac{180}{x} - 3\right)$

$= 180 - \frac{360}{x} - 3x + 6$

$\boxed{A(x) = 186 - 3x - 360x^{-1}}$

$x - 2 \geq 0 \to x \geq 2$

$y - 3 \geq 0 \to y \geq 3 \to \frac{180}{x} \geq 3 \to 60 \geq x$

so $\boxed{2 \leq x \leq 60}$.

$A'(x) = -3 - 360(-x^{-2}) = -3 + 360/x^2 = 0$

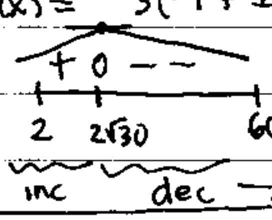
$A''(x) = 0 + 360(-2x^{-3}) = -2\cdot 360x^{-3} < 0$ $\frown$

$-1 + \frac{120}{x^2} = 0 \quad \frac{120}{x^2} = 1 \quad x^2 = 120 = 4\cdot 30$

$x = \sqrt{4\cdot 30} = 2\sqrt{30} \approx 10.95$ crit value $\frown$ local max.

$y = \frac{180}{2\sqrt{30}} = \frac{6\cdot 30}{2\sqrt{30}} = 3\sqrt{30} \approx 16.43$ first der test:

$A'(x) = 3\left(-1 + \frac{120}{x^2}\right) = \frac{3(120 - x^2)}{x^2}$

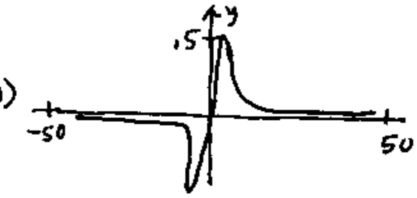


local max by 1st der test.

→ must be global max

$\boxed{\text{poster should have horizontal width } 2\sqrt{30} \approx 10.95 \text{ in and height } 3\sqrt{30} \approx 16.43 \text{ in.}}$

⑨ a)



appears that $y=0$ is a horasympt.

```
> seq(f(x), x = [10., 100., 1000., 10000.]);
```

$.068, .0067, .00067, .000067 \to 0^+$

```
seq(f(x), x= [-10., -100., -1000., -10000.]);
```

$-.067, -.0067, -.00067, -.00067 \to 0^-$

suggests $\lim\limits_{x\to\infty} f(x) = 0 = \lim\limits_{x\to-\infty} f(x)$.

b) $f'(x) = \frac{d}{dx}\arctan(3x) - \frac{d}{dx}\arctan x$

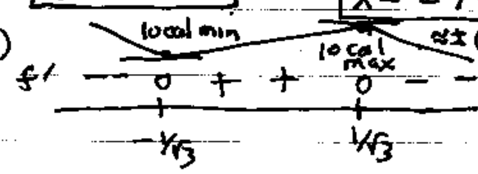
$= \frac{1}{1+(3x)^2}\frac{d}{dx}(3x) - \frac{1}{1+x^2} = \frac{3}{1+9x^2} - \frac{1}{1+x^2}$

$= \frac{3(1+x^2) - (1+9x^2)}{(1+x^2)(1+9x^2)} = \frac{3+3x^2-1-9x^2}{(\;)(\;)} = \frac{2-6x^2}{(\;)(\;)}$

$= \boxed{\frac{2(1-3x^2)}{(1+x^2)(1+9x^2)}} = 0 \to 1 - 3x^2 = 0$ $\quad x^2 = \frac{1}{3}$

$\boxed{x = \pm 1/\sqrt{3}}$ $\approx \pm 0.5776$ two critical points

c)



d) $0 = f''(x) \to \boxed{x=0}$, $-13 - 18x^2 + 27x^4 = 0$

could do quad form: $x^2 = \frac{18 \pm \sqrt{18^2 - 4(27)(-13)}}{2\cdot 27}$

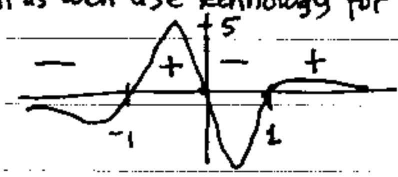
then only + root is positive

$x = \pm(x_+)^{1/2}$ ... but only numerical value needed:

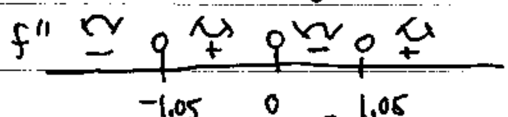
```
> fsolve(-13-18*x^2+27*x^4, x);
```

$\pm 1.05030175 \approx \boxed{\pm 1.0503 = x}$

might as well use technology for sign f'' too:



all three are pts of inflection



e) $f\left(\frac{1}{\sqrt{3}}\right) = \arctan\left(\frac{3}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)$ "$\sqrt{3}$"

60° 30°

$= \pi/3 - \pi/6 = \pi/6 \approx 0.5236$

$f(-1/\sqrt{3}) = -\pi/6$ odd function since tan, arctan are odd functions.

crits: $\boxed{(1/\sqrt{3}, \pi/6), (-1/\sqrt{3}, -\pi/6)}$

use technology for numerical roots.

points of inflection: $\boxed{(0,0)}$

$\boxed{(1.0503, 0.4536), (-1.0503, -0.4536)}$

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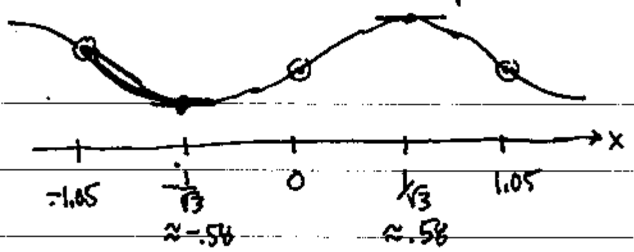
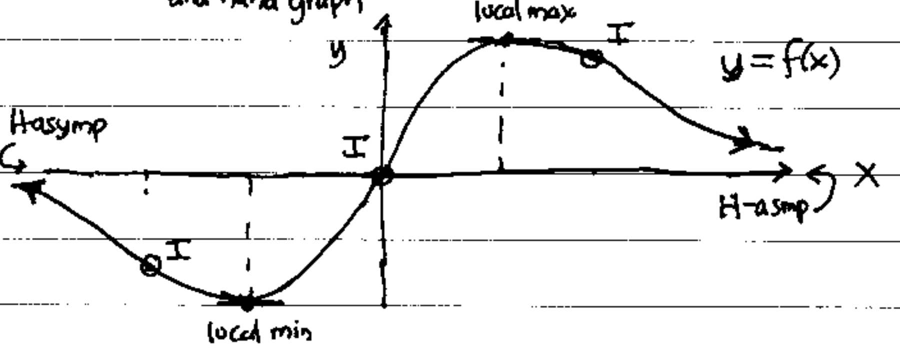
9 f) odd function, 2 crits, 3 pts of inflection



curved stick figure plot. now plot 5 pts and hand graph



7b) $\frac{d}{dt}\left[x^{1/2}+y^{1/2}=3\right]$ $\quad \frac{dy}{dt}=-2 \quad \left.\frac{dx}{dt}\right|_{x=4}=?$

$\frac{1}{2}x^{-1/2}\frac{dx}{dt}+\frac{1}{2}y^{-1/2}\frac{dy}{dt}=0$

$\frac{dx}{dt}=-\frac{x^{1/2}}{y^{1/2}}\frac{dy}{dt}$

$x=4\to y=1 \quad \left(4^{1/2}+y^{1/2}=3\to y^{1/2}=3-2=1 \right.$
$\left.\to y=1 \text{ since } y>0\right)$

$\left.\frac{dx}{dt}\right|_{x=4}=-\frac{4^{1/2}}{1^{1/2}}(-2)=+4$

$\boxed{\text{x is increasing at } 4 \text{ cm/sec}}$

8b) $x+4y=1000 \qquad x>0,\ y>0,$ integers

max $P=xy=(1000-4y)y \quad >0$

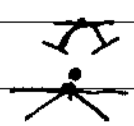
$x=1000-4y \qquad$ since $x>0$

$P(y)=1000y-4y^2 \qquad 4y<1000$

on $0<y<250 \qquad y<250$

$P'(y)=1000-8y=0\to y=\frac{1000}{8}=125$

$P''(y)=-8<0$ $\to x=1000-4(125)$
$=1000-500=500$

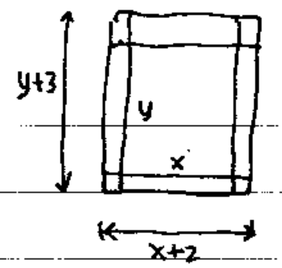


2nd der test local max

1st der test → inc then dec, global max

$\boxed{\text{The two positive integers are } 125 \text{ and } 500}$

8) alternate solution using internal dimensions



$(x+2)(y+3)=180$

$\to y+3=180/(x+2)$

$y=\frac{180}{x+2}-3$

max: $A=xy=x\left(\frac{180}{x+2}-3\right)$

$A(x)=180\left(\frac{x}{x+2}\right)-3x$

$A'(x)=180\,\frac{(x+2)1-x\cdot 1}{(x+2)^2}-3$ (better not to combine first → much longer algebra)

$=\frac{360}{(x+2)^2}-3=0\to$

$\frac{360}{(x+2)^2}=3\to (x+2)^2=\frac{360}{3}=120$

$x+2=\sqrt{120}\to x=\sqrt{120}-2$
$=2\sqrt{30}-2$

$y=\frac{180}{\sqrt{120}}-3=\frac{6\cdot 30}{2\sqrt{30}}-3=3\sqrt{30}-3$

but the most natural dimensions to give for the poster are the outer dimensions

[think: 8½ x 11 inch paper not 6½ x 9 (with usual 1 inch margins]

9 a) afterthought: symbolically:

$\lim_{x\to\infty}\arctan x=\frac{\pi}{2}$

$\lim_{x\to\infty}\arctan 3x=\frac{\pi}{2}$

$\to \lim_{x\to\pm\infty} f(x)=\frac{\pi}{2}-\frac{\pi}{2}=0.$