MAT 1505-05/08 01S Quiz 10 Homework Assignment

① $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ Taylor expand about $x=1$ and then about $x=2$.
Check with MAPLE and print out your verification.

② Stewart 11.R.58: The force due to gravity on an object with mass m at a height h above the surface of the Earth is $F = \frac{mgR^2}{(R+h)^2}$, where R is the radius of Earth and g is the acceleration due to gravity.

a) Express F as a series in powers of h/R.

b) Observe that if we approximate F by the first term in the series, we get the expression $F \approx mg$ that is usually used when h is much smaller than R. Use the Alternating Series Estimation Theorem to estimate the range of values of h for which the approximation $F \approx mg$ is accurate to within 1%. (Use $R = 6400$ km)

then backsubstitute $x = h/R$.

Hint: Use $\frac{1}{(R+h)^2} = R^{-2}(1+x)^{-2}$ with $x = h/R$. First Taylor expand $f(x) = (1+x)^{-2}$ and easily get formula for nth term

① a)

$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$	$f(1) = 3-4-12+5 = -8$	$f(2) = 3\cdot 2^4 - 4\cdot 2^3 - 12\cdot 2^2 + 5 = -27$
$f'(x) = 12x^3 - 12x^2 - 24x$	$f'(1) = 12-12-24 = -24$	$f'(2) = 12\cdot 2^3 - 12\cdot 2^2 - 24\cdot 2 = 0$
$f''(x) = 36x^2 - 24x - 24$	$f''(1) = 36-24-24 = -12$	$f''(2) = 36\cdot 2^2 - 24\cdot 2 - 24 = 72$
$f^{(3)}(x) = 72x - 24$	$f^{(3)}(1) = 72-24 = 48$	$f^{(3)}(2) = 72\cdot 2 - 24 = 120$
$f^{(4)}(x) = 72$	$f^{(4)}(1) = 72$	$f^{(4)}(2) = 72$
$f^{(5)}(x) = 0$		

f:=x->3*x^4-4*x^3-12*x^2+5;

$$f(x) = \sum_{n=0}^{5} \frac{f^{(n)}(1)(x-1)^n}{n!} = -8 - \frac{24}{1!}(x-1) - \frac{12}{2!}(x-1)^2 + \frac{48}{3!}(x-1)^3 + \frac{72}{4!}(x-1)^4$$
$$= \boxed{-8 - 24(x-1) - 6(x-1)^2 + 8(x-1)^3 + 3(x-1)^4}$$

> taylor(f(x), x=1, 5);

$$f(x) = \sum_{n=0}^{5} \frac{f^{(n)}(2)(x-2)^n}{n!} = -27 + \frac{72}{2!}(x-2)^2 + \frac{120}{3!}(x-2)^3 + \frac{72}{4!}(x-2)^4$$
$$= \boxed{-27 + 36(x-2)^2 + 20(x-2)^3 + 3(x-2)^4}$$

> taylor(f(x), x=2, 5);

②
$$F = \frac{mgR^2}{(R+h)^2} = \frac{mgR^2}{(R(1+\frac{h}{R}))^2} = \frac{mg\cancel{R^2}}{\cancel{R^2}(1+\frac{h}{R})^2} = mg\left(1+\frac{h}{R}\right)^{-2} = mg\underbrace{(1+x)^{-2}}_{f(x)} \qquad x = \frac{h}{R}$$

$f(x) = (1+x)^{-2}$	$f(0) = 1$	$= 1!$
$f'(x) = (-2)(1+x)^{-3}$	$f'(0) = (-2)$	$= -2!$
$f^{(2)}(x) = (-2)(-3)(1+x)^{-4}$	$f''(0) = (-2)(-3)$	$= +3!$
$f^{(3)}(x) = (-2)(-3)(-4)(1+x)^{-5}$	$f^{(3)}(0) = (-2)(-3)(-4)$	$= -4!$

$$f^{(n)}(0) = (-1)^n (n+1)!$$

$(n+1)\,n!$

$$\frac{f^{(n)}(0)}{n!} = (-1)^n \frac{(n+1)!}{n!} = (-1)^n (n+1)\frac{n!}{n!} = (-1)^n (n+1)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$
$$= 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$F = mg \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{h}{R}\right)^n = mg\left(1 - 2\left(\frac{h}{R}\right) + 3\left(\frac{h}{R}\right)^2 - \dots\right) = mg - \underbrace{\frac{2mgh}{R}}_{\text{estimate of maximum error in using first term alone}} + \dots$$

$\frac{2mgh}{R} \le .01\, mg$ (error less than 1% of mg)

$\frac{2h}{R} \le .01 \qquad h \le \frac{.01}{2} R = .005R = .005(6400)\text{ km} = \boxed{32\text{ km}}$

$$\boxed{h \le 32\text{ km}}$$