MAT1505-05/08 01S Takehome Test 3 answers

① a) $f(x)=\ln x$

$f'(x)=\frac{1}{x}=x^{-1}$

$f''(x)=(-1)x^{-2}$

$f'''(x)=(-1)(-2)x^{-3}$

$f^{(4)}(x)=(-1)(-2)(-3)x^{-4}$

$f^{(n)}(x)=(-1)^{n-1}(n-1)!\,x^{-n},\ n\ge 1$

$f^{(n)}(1)=(-1)^{n-1}(n-1)!\,1^{-n}$

$=(-1)^{n-1}(n-1)!\quad n\ge 1$

$f(1)=\ln 1=0$

$\frac{f^{(n)}(1)}{n!}=(-1)^{n-1}\frac{(n-1)!}{n!}=(-1)^{n}\frac{1}{n}$

$\sum_{n=0}^{\infty}\frac{f^{(n)}(1)(x-1)^n}{n!}=\underbrace{f(1)}_{0}+\sum_{n=1}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n$

$=\boxed{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}=\ln x}$

b) $=(x-1)-\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}+\frac{(x-1)^5}{5}-\dots$

even if you just wrote out the first few terms you can guess the general result

c) $x=1.1:,\ x-1=.1:$

$\ln 1.1=.1-\frac{.01}{2}+\frac{10^{-3}}{3}-\frac{10^{-4}}{4}+\frac{10^{-5}}{5}-\dots$

$=.10000$
$-.00500$
$+.00033$
$-.000025 \leftarrow$ abs val $<.5\times10^{-4}$
$+.000002$
$-\dots$

$S_4\approx .09533\to\boxed{.0953}$

② a) $\int\frac{1}{x\ln x}dx=\int\frac{1}{u}du=\ln|u|+C$

$=\boxed{\ln|\ln x|+C}$

$u=\ln x$

$\frac{du}{dx}=\frac{1}{x}$

$du=\frac{1}{x}dx$

$\int\frac{1}{x(\ln x)^p}dx=\int\frac{1}{u^p}du=\int u^{-p}du$

$=\frac{u^{1-p}}{1-p}+C=\boxed{\frac{(\ln x)^{1-p}}{1-p}+C}\quad p>1$

b) Integral test $p\neq1$: $\int_2^\infty\frac{1}{x\ln x}dx=\lim_{a\to\infty}\ln|\ln x|\Big|_2^a$

$=\lim_{a\to\infty}\ln(\ln a)-\ln(\ln 2)=\infty$ diverges so series diverges

②b) $p>1$: $\int_2^\infty\frac{1}{x(\ln x)^p}dx=\lim_{a\to\infty}\frac{(\ln x)^{1-p}}{1-p}\Big|_2^a=\lim_{a\to\infty}\underbrace{\frac{(\ln a)^{1-p}}{1-p}}_{\to0}-\frac{(\ln 2)^{1-p}}{1-p}$ negative

$=\frac{(\ln 2)^{1-p}}{p-1}$ converges so series converges.

conclusion: $\boxed{\text{series converges only for } p>1 \text{ and not } p=1}$.

③ a) $\int xe^{-2x^2}dx=\int e^u\left(-\frac{du}{4}\right)=-\frac{1}{4}\int e^u du=-\frac{1}{4}e^u+C$

$=-\frac{1}{4}e^{-2x^2}+C.$

$u=-2x^2$

$\frac{du}{dx}=-4x$

$du=-4x\,dx$

$-\frac{du}{4}=x\,dx$

$\int_0^a xe^{-2x^2}dx=-\frac{1}{4}e^{-2x^2}\Big|_0^a$

$=-\frac{1}{4}e^{-2a^2}+\frac{1}{4}e^{-0}=\boxed{\frac{1}{4}-\frac{1}{4}e^{-2a^2}}$

$\int_0^\infty xe^{-2x^2}dx=\lim_{a\to\infty}\left(\frac{1}{4}-\frac{1}{4}\underbrace{e^{-2a^2}}_{\to0}\right)=\boxed{\frac{1}{4}}$

b) $\int\underbrace{x}_{u}\underbrace{e^{-2x}dx}_{dv}=\underbrace{x}_{u}\underbrace{\left(-\frac{1}{2}e^{-2x}\right)}_{v}-\int\underbrace{\left(-\frac{1}{2}e^{-2x}\right)}_{v}\underbrace{dx}_{du}$

$du=dx\quad dv=e^{-2x}$

$v=\int e^{-2x}dx$

$=-\frac{1}{2}e^{-2x}$

$=-\frac{x}{2}e^{-2x}+\frac{1}{2}\int e^{-2x}dx$

$=\boxed{-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}+C}$

$\int_0^a xe^{-2x}dx=-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}\Big|_0^a=-\frac{a}{2}e^{-2a}-\frac{1}{4}e^{-2a}+0+\frac{1}{4}e^0$

$=\boxed{\frac{1}{4}-\frac{a}{2}e^{-2a}-\frac{1}{4}e^{-2a}}$

$\int_0^\infty xe^{-2x}dx=\lim_{a\to\infty}\left(\frac{1}{4}-\underbrace{\frac{a}{2e^{2a}}}_{0}-\frac{1}{4}\underbrace{e^{-2a}}_{\to0}\right)=\boxed{\frac{1}{4}}$

$\lim_{a\to\infty}\frac{a\to\infty}{e^{2a}\to\infty}=\lim_{a\to\infty}\frac{\frac{d}{da}(a)}{\frac{d}{da}(e^{2a})}=\lim_{a\to\infty}\frac{1}{2e^{2a}}=0$

④ b) $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n\quad(x\to\frac{x}{2})\quad\frac{1}{1-\frac{x}{2}}=\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n=\sum_{n=0}^{\infty}\frac{x^n}{2^n}$

ratio $\left|\frac{x}{2}\right|<1\to|x|<2\to\boxed{-2<x<2}$

a) $f(x)=(1-\frac{x}{2})^{-1},\ g(x)=xf'(x)=x(-1)(1-\frac{x}{2})^{-2}(-\frac{1}{2})$

$=\boxed{\frac{x}{2}(1-\frac{x}{2})^{-2}=\frac{x/2}{(1-x/2)^2}}$

c) $g(x)=x\sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{x^n}{2^n}\right)=\sum_{n=0}^{\infty}x\cdot\frac{nx^{n-1}}{2^n}=\sum_{n=0}^{\infty}\frac{nx^n}{2^n}$

$=\boxed{\sum_{n=1}^{\infty}\frac{nx^n}{2^n}}$ (since first term is zero) (convergence at end)

d) $g(1)=\frac{\frac{1}{2}}{(1-\frac{1}{2})^2}=\frac{\frac{1}{2}}{(\frac{1}{2})^2}=2$

$=\sum_{n=1}^{\infty}\frac{n\cdot1^n}{2^n}=\sum_{n=1}^{\infty}\frac{n}{2^n}$

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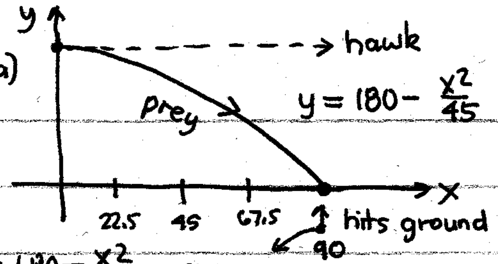
⑤ a)



$y = 180 - \frac{x^2}{45} = 0$

$x^2 = 45 \cdot 180 = 45 \cdot 2 \cdot 90 = 90^2$

$x = 90$ (since $x > 0$)

$$s = \int_0^{90} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{90} \sqrt{1 + \left(-\frac{2x}{45}\right)^2}\, dx = \boxed{\int_0^{90} \sqrt{1 + \frac{4x^2}{45^2}}\, dx}$$

b) $\Delta x = \frac{90}{4} = 22.5$

$$S_4 = \frac{\Delta x}{3}\left[f(0) + 4f(22.5) + 2f(45) + 4f(67.5) + f(90)\right]$$

$$= \frac{22.5}{3}\left[1.00.. + 5.657.. + 4.472.. + 12.650.. + 4.123...\right]$$

$$= 209.259 \approx \boxed{209.3} \approx \boxed{209}$$

(not bad compared to exact value ≈ 209.105)

⑥ $\int_0^{24} c(t)\,dt \to$ 12 intervals pairs in data $\to n = 12$

$\Delta t = \frac{24}{12} = 2$ interval width

$$S_{12} = \frac{1}{3}(2)\left[c(0) + 4c(2) + 2c(4) + 4c(6) + 2c(8) + 4c(10) + 2c(12) + 4c(14) + 2c(16) + 4c(18) + 2c(20) + 4c(22) + c(24)\right]$$

$$= \frac{2}{3}\left[0 + 4(1.9) + 2(3.3) + 4(5.1) + 2(7.6) + 4(7.1) + 2(5.8) + 4(4.7) + 2(3.3) + 4(2.1) + 2(1.1) + 4(0.5) + 0\right]$$

$$= 83.7333. \approx \boxed{84}$$ (2 significant digits)

⑦ $f_{avg} = \frac{1}{2-0}\int_0^2 f(x)\,dx = \frac{1}{2}\int_0^2 x^2(1+x^3)^{1/2}\,dx$

$u = 1 + x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\,dx$

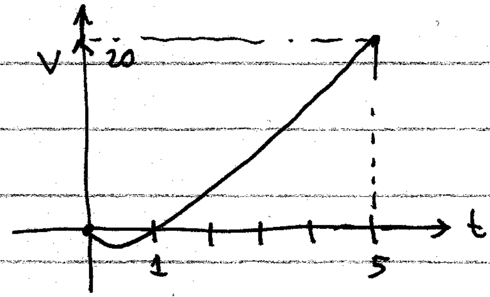
$\frac{du}{3} = x^2\,dx$

$$= \frac{1}{2}\int_{x=0}^{x=2} u^{1/2}\left(\frac{du}{3}\right) = \frac{1}{6}\int_{x=0}^{x=2} u^{1/2}\,du$$

$$= \frac{1}{6}\cdot\frac{2}{3}u^{3/2}\Big|_{x=0}^{x=2} = \frac{1}{9}(1+x^3)^{3/2}\Big|_0^2$$

$$= \frac{1}{9}(9^{3/2} - 1) = \frac{1}{9}(27-1) = \boxed{\frac{26}{9}}$$

⑧ $V(t) = t^2 - t \qquad t \in [0,5]$



a) displacement: $S(5) = \int_0^5 V(t)\,dt = \int_0^5 t^2 - t\,dt$

$$= \frac{t^3}{3} - \frac{t^2}{2}\Big|_0^5 = \frac{5^3}{3} - \frac{5^2}{2} = 5^2\left(\frac{5}{3} - \frac{1}{2}\right)$$

$$= \frac{7}{6}5^2 = \boxed{\frac{175}{6} \approx 29\tfrac{1}{6}}$$

b) distance traveled:

$$d(5) = \int_0^5 |V(t)|\,dt = \int_0^1 -V(t)\,dt + \int_1^5 V(t)\,dt$$

$$= \int_0^1 (t - t^2)\,dt + \int_1^5 (t^2 - t)\,dt$$

$$= \left(\frac{t^2}{2} - \frac{t^3}{3}\right)\Big|_0^1 + \left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_1^5$$

$$= \frac{1}{2} - \frac{1}{3} + \left(\frac{5^3}{3} - \frac{5^2}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$+ \left(\frac{1}{2} - \frac{1}{3}\right)$

$$= 2\left(\frac{1}{6}\right) + \left(\frac{5^3}{3} - \frac{5^2}{2}\right)$$

$$= \frac{1}{3} + \frac{175}{6} = \boxed{\frac{177}{6} = 29\tfrac{1}{2}}$$

(also interpretable as area between t axis and graph between 0 and 5)

④ c) interval of convergence

$$g(x) = \sum_{n=1}^{\infty} \underbrace{\frac{nx^n}{2^n}}_{a_n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{2^{n+1}} \cdot \frac{2^n}{n|x|^n}$$

$$= \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \frac{|x|^{n+1}}{|x|^n}\left(\frac{n+1}{n}\right) = \frac{|x|}{2}\cdot 1 = \frac{|x|}{2} < 1$$

$|x| < 2$, $-2 < x < 2$.

endpoints:

$$x = \pm 2: \quad \sum_{n=1}^{\infty} \frac{n(\pm 2)^n}{2^n} = \sum_{n=1}^{\infty} (\pm 1)^n n \frac{2^n}{2^n} = \sum_{n=1}^{\infty} (\pm 1)^n n$$

both diverge so valid only for $-2 < x < 2$.