Mat2705 005 Final Exam Answers (1)

① a) (i) $3\frac{dx}{dt}+2x=e^{-t/2}$

$$e^{2/3\,t}\left[\frac{dx}{dt}+\underbrace{\tfrac{2}{3}}\,x=\tfrac{1}{3}e^{-t/2}\right]$$

$$e^{\int\frac{2}{3}dt}=e^{\frac{2}{3}t}$$

$$\left(xe^{\frac{2t}{3}}\right)'=\tfrac{1}{3}e^{\frac{2}{3}t}e^{-\frac{1}{2}t}=\tfrac{1}{3}e^{t/6}$$

$$xe^{2t/3}=\tfrac{1}{3}\,\frac{e^{t/6}}{1/6}+C_1$$

$$\boxed{x=e^{-2/3\,t}\left[2e^{t/6}+C_1\right]=2e^{-t/2}+C_1e^{-2/3\,t}}$$

(ii) $x(0)=2+C_1=0\rightarrow C_1=-2$

$$\boxed{x=2e^{-t/2}-2e^{-2/3\,t}}$$

b) (i) $\frac{dy}{dt}=\frac{y+1}{t+1}$

$$\int\frac{dy}{y+1}=\int\frac{dt}{t+1}$$

$$\ln|y+1|=\ln|t+1|+C_1$$

$$e \qquad e$$

$$|y+1|=e^{C_1}|t+1|$$

$$y+1=\underbrace{\pm e^{C_1}}_{C_2}(t+1)$$

$$\boxed{y=-1+C_2(t+1)}$$

(ii) $y(0)=-1+C_2=\begin{Bmatrix}0\\1\\-2\end{Bmatrix}$

$$\rightarrow C_2=1+\begin{Bmatrix}0\\1\\-2\end{Bmatrix}=\begin{Bmatrix}1\\2\\-1\end{Bmatrix}$$

bob: $y(t)=-1+1(t+1)=\boxed{t}$ ✓

chuck: $y(t)=-1+2(t+1)=\boxed{2t+1}$ ✓

dave: $y(t)=-1+(-1)(t+1)=\boxed{-t-2}$ ⊗

(iii) bob and chuck are right, dave is wrong.

(iv) $y=t$: $\frac{dy}{dt}=1$, $\frac{y+1}{t+1}=\frac{t+1}{t+1}=1$ ✓

② a) $y''+3y'+2y=e^{-t/2}$

$y=e^{rt}$: $(r^2+3r+2)e^{rt}=0$ (hom soln)

$r^2+3r+2=0$, $(r+1)(r+2)=0$, $r=-1,-2$

$y_h=c_1e^{-t}+c_2e^{-2t}$

$2\,[\,y_p=Ae^{-t/2}\,]$

$3\,[\,y_p'=-\frac{A}{2}e^{-t/2}\,]$

$1\,[\,y_p''=\frac{A}{4}e^{-t/2}\,]$

$$y_p''+3y_p'+2y_p=\left(\tfrac{1}{4}-\tfrac{3}{2}+2\right)Ae^{-t/2}=\tfrac{3}{4}Ae^{-t/2}=e^{-t/2}$$

$\therefore \frac{3A}{4}=1$, $A=4/3$, $y_p=4/3\,e^{-t/2}$

$$\boxed{y=y_h+y_p=c_1e^{-t}+c_2e^{-2t}+4/3\,e^{-t/2}}$$

b) $y(0)=c_1+c_2+4/3=0$

$y'(t)=-c_1e^{-t}-2c_2e^{-2t}-\frac{2}{3}e^{-t/2}$

$y'(0)=-c_1-2c_2-2/3=0$

$$\begin{bmatrix}1&1\\-1&-2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-4/3\\2/3\end{bmatrix}\qquad \begin{bmatrix}c_1\\c_2\end{bmatrix}=\frac{1}{-1}\begin{bmatrix}-2&-1\\1&1\end{bmatrix}\begin{bmatrix}-4/3\\2/3\end{bmatrix}=\begin{bmatrix}-6/3\\2/3\end{bmatrix}\ {}^{\nearrow -2}$$

$$\boxed{y=-2e^{-t}+\tfrac{2}{3}e^{-2t}+\tfrac{4}{3}e^{-t/2}}$$

③ a)

$$\begin{bmatrix}3&0&1&0\\0&3&1&0\\0&0&2&0\\0&0&0&2\end{bmatrix}\begin{bmatrix}1\\2\\3\\0\end{bmatrix}=\begin{bmatrix}3+3\\6+3\\6\\0\end{bmatrix}=\begin{bmatrix}6\\9\\6\\0\end{bmatrix}\neq\lambda\begin{bmatrix}1\\2\\3\\0\end{bmatrix}$$ not an eigenvector

$$\begin{bmatrix}3&0&1&0\\0&3&1&0\\0&0&2&0\\0&0&0&2\end{bmatrix}\begin{bmatrix}3\\3\\-3\\-2\end{bmatrix}=\begin{bmatrix}9-3\\9-3\\-6\\-4\end{bmatrix}=\begin{bmatrix}6\\6\\-6\\-4\end{bmatrix}=2\begin{bmatrix}3\\3\\-3\\-2\end{bmatrix}$$ eigenvector with eigenvalue 2

b) $\lambda=2$: $(A-\lambda I)=\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$ already reduced

RREF
$$\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\qquad \begin{aligned}x_1+x_3&=0\\x_2+x_3&=0\\0&=0\\0&=0\end{aligned}$$

$x_1=-x_3=-t_1$

$x_2=-x_3=-t_1$

$x_3=t_1$

$x_4=t_2$

$\vec{x}=(-t_1,-t_1,t_1,t_2)$

$=t_1(-1,-1,1)+t_2(0,0,0,1)$

basis of eigenspace: $\boxed{\{(-1,-1,1),(0,0,0,1)\}}$

dimension is $\boxed{\text{two}}$ (# of basis vectors)

Mat 2705 005 Final Exam Answers Page 2

④ a) $\vec{x}' = \underset{3\times3}{A}\,\vec{x}$ (3×1) : $\boxed{\begin{array}{ll} x_1' = -8x_1 - 12x_2 - 6x_3 , & x_1(0)=1 \\ x_2' = 2x_1 + x_2 + 2x_3 , & x_2(0)=0 \\ x_3' = 7x_1 + 12x_2 + 5x_3 , & x_3(0)=0 \end{array}}$

$\vec{x}(0) = (1,0,0)$

$\vec{x} = B\vec{y}$, where $B = \text{augment}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} -1 & -6 & -4 \\ 0 & 1 & 1 \\ 1 & 5 & 4 \end{bmatrix}$ decouples the system:

$\vec{x}' = A\vec{x} \to (B\vec{y})' = A(B\vec{y}) \to B^{-1}B\vec{y}' = (B^{-1}AB)\vec{y} \to \vec{y}' = \text{diag}(-2,-1,1)\vec{y}$

$y_1' = -2y_1 \to y_1 = c_1 e^{-2t} \to \vec{x} = \begin{bmatrix} -1 & -6 & -4 \\ 0 & 1 & 1 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{-t} \\ c_3 e^{t} \end{bmatrix} = \boxed{\begin{bmatrix} -c_1 e^{-2t} - 6c_2 e^{-t} - 4c_3 e^{t} \\ c_2 e^{-t} + c_3 e^{t} \\ c_1 e^{-2t} + 5c_2 e^{-t} + 4c_3 e^{t} \end{bmatrix} = \vec{x}}$

$y_2' = -1\,y_2 \qquad y_2 = c_2 e^{-t}$

$y_3' = 1\,y_3 \qquad y_3 = c_3 e^{t}$

or write out directly eigenvalue exponential/eigenvector soln:

$\vec{x} = c_1 e^{-2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix} + c_3 e^{t} \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} = $ combine

b) $\vec{x}(0) = \begin{bmatrix} -1 & -6 & -4 \\ 0 & 1 & 1 \\ 1 & 5 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} -1 & -6 & -4 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 5 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 5 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 & -1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & -1 & 0 & +1 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 4 & -1 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 1 & +1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \boxed{1} & +1 \end{bmatrix}$

$\to \begin{bmatrix} 1 & 0 & 0 & +1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & +1 \end{bmatrix} \quad \begin{array}{l} c_1 = 1 \\ c_2 = -1 \\ c_3 = 1 \end{array}$ so $\boxed{\begin{array}{l} x_1 = -e^{-2t} + 6e^{-t} - 4e^{t} \\ x_2 = \quad -e^{-t} + e^{t} \\ x_3 = e^{-2t} - 5e^{-t} + 4e^{t} \end{array}}$

⑤ a) $|A - \lambda I| = \begin{vmatrix} -2-\lambda & -5/2 \\ 10 & -2-\lambda \end{vmatrix} = (\lambda+2)^2 + \underbrace{\tfrac{5}{2}(10)}_{25} = \lambda^2 + 4\lambda + 4 + 25 = \lambda^2 + 4\lambda + 29 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{16 - 4 \cdot 29}}{2} = \dfrac{-4 \pm \sqrt{4(4-29)}}{2} = -2 \pm 5i$

$\lambda = -2+5i$: $A - \lambda I = \begin{bmatrix} -2-(-2+5i) & -5/2 \\ 10 & -2-(-2+5i) \end{bmatrix} = \begin{bmatrix} -5i & -5/2 \\ 10 & -5i \end{bmatrix} \to \begin{bmatrix} 1 & -i/2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} x_1 - \frac{i}{2}x_2 = 0 \\ 0 = 0 \end{array}$

B F

$x_1 = \frac{i}{2}t \quad \vec{x} = (\tfrac{i}{2}t, t) = t\underbrace{(\tfrac{i}{2}, \tfrac{2}{2})}$

$x_2 = t$

$\to (i,2) = \vec{b}_1 , \vec{b}_2 = (-i,2)$

eigenvalues $\boxed{\lambda_1 = -2+5i , \lambda_2 = -2-5i}$

eigenvectors $\boxed{\vec{b}_1 = (i,2) , \vec{b}_2 = (-i,2)}$

b) $e^{(-2+5i)t} \begin{bmatrix} i \\ 2 \end{bmatrix} = e^{-2t}(\cos 5t + i \sin 5t) \begin{bmatrix} i \\ 2 \end{bmatrix} = e^{-2t} \begin{bmatrix} -\sin 5t + i\cos 5t \\ 2\cos 5t + i\,2\sin 5t \end{bmatrix}$

$= \underbrace{e^{-2t} \begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix}}_{\vec{x}_1} + i \underbrace{e^{-2t} \begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix}}_{\vec{x}_2}$ c) so $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 e^{-2t} \begin{bmatrix} -\sin 5t \\ 2\cos 5t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} \cos 5t \\ 2\sin 5t \end{bmatrix}$

$\boxed{\vec{x} = \begin{bmatrix} e^{-2t}(-c_1 \sin 5t + c_2 \cos 5t) \\ e^{-2t}(2c_1 \cos 5t + 2c_2 \sin 5t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$

d) $\vec{x}(0) = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_1 \end{bmatrix} \to \begin{array}{l} c_2 = 3 \\ c_1 = 3/2 \end{array}$

$\boxed{\begin{array}{l} x_1 = e^{-2t}(-\frac{3}{2}\sin 5t + 3\cos 5t) \\ x_2 = e^{-2t}(3\cos 5t + 6\sin 5t) \end{array}}$