MAT 2705 TEST 1 Answers

① a) $\frac{dy}{dx} = x^2 y^{2/3}$ separable

$y^{-2/3}\,dy = x^2\,dx$ separate

$\int y^{-2/3}\,dy = \int x^2\,dx$ integrate

$\frac{y^{1/3}}{1/3} = \frac{x^3}{3} + C_1$

$y^{1/3} = \frac{1}{9}x^3 + \frac{1}{3}C_1$ solve for y

$$\boxed{y = \left(\frac{1}{9}x^3 + \frac{1}{3}C_1\right)^3 = \left(\frac{1}{9}x^3 + C_2\right)^3}$$

b) $\frac{dy}{dx} = \frac{3}{x^2} - \frac{2y}{x}$ linear

$\left[\frac{dy}{dx} + \underbrace{\frac{2}{x}}\,y = \frac{3}{x^2}\right]$ standard form

$e^{\int \frac{2}{x}dx = 2\ln x} = e^{2\ln x} = (e^{\ln x})^2 = x^2$

integrating factor I.F.

$\underbrace{x^2\left[\frac{dy}{dx} + \frac{2}{x}y\right]} = x^2 \cdot \frac{3}{x^2}$ · multiply by I.F.

$\frac{d}{dx}(yx^2) = 3$

integrate

$yx^2 = \int 3\,dx = 3x + C_1$

$$\boxed{y = \frac{3x + C_1}{x^2} = \frac{3}{x} + \frac{C_1}{x^2}}$$ solve for y

c) a) $1 = y(3) = \left(\frac{1}{9}3^3 + C_2\right)^3 = (3 + C_2)^3$

$1 = 3 + C_2 \to C_2 = 1 - 3 = -2$

$$\boxed{y = \left(\frac{1}{9}x^3 - 2\right)^3}$$

b) $1 = y(3) = \frac{3(3) + C_1}{3^2} = \frac{9 + C_1}{9}$

$9 = 9 + C_1 \to C_1 = 0$

$$\boxed{y = \frac{3}{x}}$$

② $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0$

a) $y = e^{rt}$

$(r^2 + 2r + 5)e^{rt} = 0$

$r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm \frac{\sqrt{-16}}{2} = -1 \pm \frac{4i}{2} = -1 \pm 2i$

$e^{rt} = e^{(-1 \pm 2i)t} = e^{-t}e^{\pm 2it} = e^{-t}(\cos 2t \pm i\sin 2t)$

real solns: $e^{-t}\cos 2t,\ e^{-t}\sin 2t$

gen. soln: $\boxed{y = e^{-t}(c_1\cos 2t + c_2\sin 2t)}$

b) $y' = e^{-t}(-c_1\cos 2t - c_2\sin 2t)$
$\quad + e^{-t}(-2c_1\sin 2t + 2c_2\cos 2t)$

$1 = y(0) = 1\cdot(c_1 \cdot 1 + c_2 \cdot 0) = c_1 \to \boxed{c_1 = 1}$

$3 = \frac{dy}{dt}(0) = 1(-c_1 \cdot 1 - c_2 \cdot 0) + 1(-2c_1 \cdot 0 + 2c_2 \cdot 1) = -c_1 + 2c_2$

$c_2 = \frac{3 + c_1}{2} = \frac{3+1}{2} = 2$

$$\boxed{y = e^{-t}(\cos 2t + 2\sin 2t)}$$

③ $4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 8e^{-t}$ or $\frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{1}{4}y = 2e^{-t}$

a) $y = e^{rt}$

$(4r^2 + 4r + 1)e^{rt} = 0$; $r^2 + r + \frac{1}{4} = 0$

$4r^2 + 4r + 1 = 0$; $r = \frac{-1 \pm \sqrt{1 - 4(1/4)}}{2} = -\frac{1}{2}$

$r = \frac{-4 \pm \sqrt{4^2 - 4(4)}}{2\cdot 4} = -\frac{1}{2}$

$e^{rt} = e^{-\frac{1}{2}t}$ only 1 ind soln $\to$ $te^{-\frac{1}{2}t}$ second ind soln

$y_c = (c_1 + c_2 t)e^{-\frac{1}{2}t}$

b) $8e^{-t} \to y_p = A_0 e^{-t}$ ← not a soln of hom DE

$y_p' = -A_0 e^{-t}$

$y_p'' = A_0 e^{-t}$

$4y_p'' + 4y_p' + y_p = 4(A_0e^{-t}) + 4(-A_0e^{-t}) + A_0e^{-t}$
$= A_0e^{-t} = 8e^{-t} \to A_0 = 8$

$\boxed{y_p = 8e^{-t}}$

c) $y = y_c + y_p = (c_1 + c_2 t)e^{-\frac{1}{2}t} + 8e^{-t}$

d) $y' = c_2 e^{-\frac{1}{2}t} + (c_1 + c_2 t)(-\frac{1}{2})e^{-\frac{1}{2}t} - 8e^{-t}$

$4 = y(0) = (c_1 + c_2 0)\cdot 1 + 8\cdot 1 = c_1 + 8 \to \boxed{c_1 = -4}$

$-1 = y'(0) = c_2 \cdot 1 + (c_1 + c_2 0)\cdot(-\frac{1}{2})\cdot 1 - 8\cdot 1 = c_2 - \frac{1}{2}c_1 - 8$

$c_2 = -1 + \frac{1}{2}c_1 + 8 = 7 + \frac{1}{2}(-4) = 7 - 2 = \boxed{5 = c_2}$

$$\boxed{y = (-4 + 5t)e^{-\frac{1}{2}t} + 8e^{-t}}$$