MAT1500-03/11 00F Final Exam Answers

1 a) $f(x) = (2-x^2)^3$

$f'(x) = \frac{d}{dx}(2-x^2)^3 = 3(2-x^2)^2 \frac{d}{dx}(2-x^2)$

$= 3(2-x^2)^2(0-2x) = \boxed{-6x(2-x^2)^2}$

b) $y = \frac{x}{(1-x^2)^{1/2}}$ $\quad \frac{dy}{dx} = \frac{(1-x^2)^{1/2}\frac{d}{dx}(x) - x\frac{d}{dx}(1-x^2)^{1/2}}{((1-x^2)^{1/2})^2}$

$= \frac{(1-x^2)^{1/2}\cdot 1 - x\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)}{(1-x^2)}$

$= \frac{(1-x^2)^{1/2} + \frac{x^2}{(1-x^2)^{1/2}}}{(1-x^2)} = \frac{\frac{(1-x^2)+x^2}{(1-x^2)^{1/2}}}{(1-x^2)}$

$= \boxed{\frac{1}{(1-x^2)^{3/2}}}$

c) $g(t) = \ln\left(\frac{t-1}{t+1}\right) = \ln(t-1) - \ln(t+1)$

$g'(t) = \frac{1}{t-1}(1) - \frac{1}{t+1}(1) = \frac{(t+1)-(t-1)}{(t-1)(t+1)} = \boxed{\frac{2}{(t-1)(t+1)}}$

d) $s = e^{-t}\sin 3t$ $\quad \frac{ds}{dt} = \frac{d}{dt}(e^{-t})\sin 3t + e^{-t}\frac{d}{dt}\sin 3t$

$= -e^{-t}\sin 3t + e^{-t}(\cos 3t)(3)$

$= \boxed{e^{-t}(-\sin 3t + 3\cos 3t)}$

e) $T = A\cos(\omega t+\delta)$, $\frac{dT}{dt} = A\frac{d}{dt}\cos(\omega t+\delta)$

$= A(-\sin(\omega t+\delta))\underbrace{\frac{d}{dt}(\omega t+\delta)}_{\to\ \omega}$

$= \boxed{-A\omega\sin(\omega t+\delta)}$

f) $\ln y = \ln\frac{x}{(1-x^2)^{1/2}} = \ln x - \frac{1}{2}\ln(1-x^2)$

$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\ln y) = \frac{1}{x} - \frac{1}{2}\frac{1}{(1-x^2)}(-2x) = \frac{1}{x} + \frac{x}{1-x^2}$

$= \frac{1-x^2+x^2}{x(1-x^2)} = \frac{1}{x(1-x^2)}$

$\frac{dy}{dx} = y\frac{1}{x(1-x^2)} = \frac{x}{(1-x^2)^{1/2}}\frac{1}{x(1-x^2)} = \boxed{\frac{1}{(1-x^2)^{3/2}}}$

g) $f'(x) = -6x(2-x^2)^2$

$f''(x) = -6\left[1(2-x^2)^2 + x(2)(2-x^2)^1(-2x)\right]$

$= -6(2-x^2)\left[(2-x^2) - 4x^2\right] = \boxed{-6(2-x^2)(2-5x^2)}$

2 $\quad xy^2 - x^2y = 6 \to \frac{d}{dx}(xy^2) - \frac{d}{dx}(x^2y) = 0$

$1\cdot y^2 + x\underbrace{\frac{d}{dx}(y^2)}_{2y\frac{dy}{dx}} - \left(2xy + x^2\frac{dy}{dx}\right) = 0$

$y^2 - 2xy + 2xy\frac{dy}{dx} - x^2\frac{dy}{dx} = 0$

$y^2 - 2xy + (2xy - x^2)\frac{dy}{dx} = 0 \to \frac{dy}{dx} = \frac{2xy-y^2}{2xy-x^2}$

2 continued $\left.\frac{dy}{dx}\right|_{(x,y)=(1,-2)} = \frac{2(1)(-2)-(-2)^2}{2(1)(-2)-1^2} = \frac{-4-4}{-4-1}$

$= \frac{-8}{-5} = \frac{8}{5}$

$y-(-2) = \frac{8}{5}(x-1) \to \boxed{y = \frac{8}{5}(x-1) - 2 \text{ or } y = \frac{8}{5}x - \frac{18}{5}}$

3 $f(4) = 5 \to x=4, y=5 \to$ pt $(4,5)$

a) $f'(4) = -2 \to m = -2$

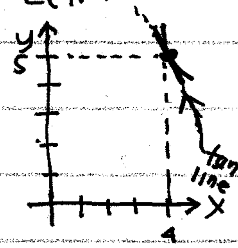
$y-5 = -2(x-4) \to \boxed{y = 5-2(x-4) \text{ or } y = 13-2x}$

b) linear approx: $L(x) = 5-2(x-4)$

$f(3.9) \approx L(3.9) = 5-2(3.9-4) = 5+.2 = \boxed{5.2}$

$f(4.1) \approx L(4.1) = 5-2(4.1-4) = 5-.2 = \boxed{4.8}$

c)



concave up: $f''(4) > 0$

d) so graph lies above tangent line, so estimates are too small.

4 $\quad V = x^3 \quad \frac{dV}{dt} = \frac{d}{dt}(x^3) = 3x^2\frac{dx}{dt} = 150$

$\frac{dx}{dt} = \frac{150}{3x^2}$, $\left.\frac{dx}{dt}\right|_{x=5} = \frac{150}{3\cdot 5^2} = \frac{150}{75} = 2$

The edgelength is increasing at 2 cm/min.

5 a) $y = \frac{x}{\sqrt{1-x^2}} \to \frac{dy}{dx} = \frac{1}{(1-x^2)^{3/2}}$ by 1 b).

$x=0 \to y = \frac{0}{\sqrt{1-0^2}} = 0$, $\frac{dy}{dx} = \frac{1}{(1-0^2)^{3/2}} = 1$

pt $(0,0)$, slope $m=1$:

$y-0 = 1(x-0) \to y = x$ linear approximation

so near $x=0$, $\frac{x}{\sqrt{1-x^2}} \approx x$.

b) $\lim_{x\to 1^-}\frac{x \to 1}{\sqrt{1-x^2} \to 0}$ determinate limit, positive so must go to $+\infty$

$= \boxed{\infty}$

6 $y = f(x) = xe^{-x} = 0 \rightleftarrows x = 0$ $\quad (0,0)$ intercept

horizontal asymptotes:

$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty}\frac{x \to \infty}{e^x \to \infty}$ "$\frac{\infty}{\infty}$" $\to$ l'Hopital's rule

$= \lim_{x\to\infty}\frac{1}{e^x \to \infty} = 0 \to y = 0$ for $x\to\infty$

$\lim_{x\to-\infty} \underset{\to -\infty}{x}\,\underset{\to\infty}{e^{-x}} = -\infty$.

critical points: $0 = f'(x) = (1-x)e^{-x} \to x = 1$

$\to y = f(1) = e^{-1}$ pt: $(1, e^{-1})$

⑥ (continued)

f': $+ + \; 0 \; - -$ $\leftarrow$ sign$(1-x)$, zero at $x = 1$

local max at $x=1$, value $e^{-1} \rightarrow$ global max since

increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$

f'': $- - \; 0 \; + +$ $\leftarrow$ sign$(x-2)$, zero at $x = 2$

concave down on $(-\infty, 2)$

concave up on $(2, \infty)$

pt of inflection at $x=2 \rightarrow y = f(2) = 2e^{-2}$.