MAT1500-03/11 00F Test 3 (Takehome) Answers

① $f(x) = \frac{1}{\sqrt{1-x}} = (1-x)^{-1/2}$

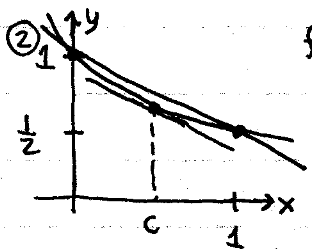
$f'(x) = -\frac{1}{2}(1-x)^{-3/2}(0-1) = \frac{1}{2(1-x)^{3/2}}$

$f(0) = \frac{1}{\sqrt{1-0}} = 1$, $f'(0) = \frac{1}{2(1-0)^{3/2}} = \frac{1}{2}$

tangent line : $(0,1)$, $m = 1/2 \to$

$y - 1 = \frac{1}{2}(x-0) \to \boxed{y = 1 + \frac{1}{2}x = L(x)}$

linear approx at $x=0$ is function whose graph is tangent line at $x=0$.

② 

$f(x) = \frac{1}{1+x}$ is continuous and differentiable on $[0,1]$ so exists c where tan line slope equals interval sec line slope:

$\frac{f(1)-f(0)}{1-0} = \frac{\frac{1}{1+1} - \frac{1}{1+0}}{1} = \frac{1}{2} - 1 = -\frac{1}{2}$

$f'(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2}$

so

$f'(c) = -\frac{1}{2} \to -\frac{1}{(1+c)^2} = -\frac{1}{2} \to (1+c)^2 = 2$

$\to 1+c = \pm\sqrt{2} \to c = -1 \pm \sqrt{2}$

only $\boxed{c = -1+\sqrt{2}} \approx .414$ lies in interval

③ 2:10 → 3:05 → 55 min $= \frac{55}{60}$ hr

$V_{avg} = \frac{74 \text{ m}}{\frac{55}{60}\text{ hr}} \approx 80.7$ mph

By Mean Value Thm, the instantaneous speed had to equal the average speed at least once during the trip, so yes the speeding ticket is justified (fortunately PA is not this advanced technologically).

④ $T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\left(\frac{L}{g}\right)^{1/2}$

$\frac{dT}{dL} = 2\pi\left(\frac{1}{2}\right)\left(\frac{L}{g}\right)^{-1/2}\left(\frac{1}{g}\right) = \frac{\pi}{g}\left(\frac{g}{L}\right)^{1/2}$

$dT = \frac{\pi}{g}\left(\frac{g}{L}\right)^{1/2} dL$

$\frac{dT}{T} = \frac{\frac{\pi}{g}\left(\frac{g}{L}\right)^{1/2} dL}{2\pi\left(\frac{L}{g}\right)^{1/2}} = \frac{1}{2g}\left(\frac{g}{L}\right)^{1} dL = \frac{1}{2}\frac{dL}{L}$

$L = 3$ (ft) $\quad dL = \pm\frac{1}{8}\left(\frac{1}{12}\right)$ (ft)

$\frac{dT}{T} = \frac{1}{2}\frac{\pm 1}{8\cdot 12\cdot 3} \approx \pm .0017 \to \approx \boxed{0.17\%}$

⑤ a) $\lim_{x\to 0} \frac{e^{ax} - e^{bx}}{x}$ "$\frac{0}{0}$" type limit so:

$= \lim_{x\to 0}\frac{\frac{d}{dx}(e^{ax} - e^{bx})}{\frac{d}{dx}x} = \lim_{x\to 0}\frac{e^{ax}(a) - e^{bx}(b)}{1} \overset{\text{determined}}{=} a - b$

b) $\lim_{x\to 1} x^{\frac{1}{1-x}}$ "1^{∞}" type limit so:

$y = x^{\frac{1}{1-x}}$, $\ln y = \frac{1}{1-x}\ln x = \frac{\ln x}{1-x}$

$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln x}{1-x}$ "$\frac{0}{0}$" type limit so:

$= \lim_{x\to 1}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}(1-x)} = \lim_{x\to 1}\frac{\frac{1}{x}}{-1} = -1$

$\lim_{x\to 1} y = \lim_{x\to 1} e^{\ln y} = e^{\lim_{x\to 1}\ln y} = \boxed{e^{-1}}$

⑥ $f(x) = (x^2-1)^3$ even since only depends on x^2

y intercept: $x = 0 \to y = f(0) = (0-1)^3 = (-1)^3 = -1$

x intercepts: $0 = y = f(x) = (x^2-1)^3 \to 0 = x^2 - 1$

$\to x^2 = 1 \to x = \pm 1$

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}(x^2-1)^3 = \infty$

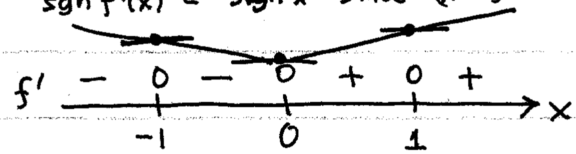
$f'(x) = \frac{d}{dx}(x^2-1)^3 = 3(x^2-1)^2(2x) = 6x(x^2-1)^2$

$f''(x) = 6\frac{d}{dx}(x(x^2-1)^2) = 6[(1)(x^2-1)^2 + x(2)(x^2-1)^1(2x)]$

$= 6[x^2-1][(x^2-1) + 4x^2] = 6(x^2-1)(5x^2-1)$

$0 = f'(x) = 6x(x^2-1)^2 \to x = 0;\ x^2 - 1 = 0 \to x = \pm 1$

sgn $f'(x)$ = sign x since $(x^2-1)^2 > 0$ if $x^2 \neq 1$.



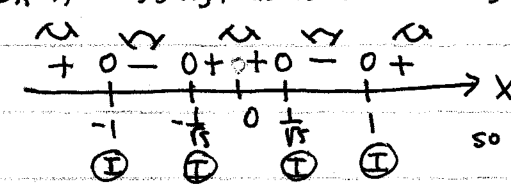
so local min at $x=0$, horizontal tangents at $x = \pm 1, 0$.

$0 = f''(x) = 6(x^2-1)(5x^2-1) \to x = \pm 1$ or $x^2 = \frac{1}{5}$

$x = \pm 1/\sqrt{5} \approx \pm .447 \to y = f(\pm 1/\sqrt{5}) = (\frac{1}{5} - 1)^3 = (-\frac{4}{5})^3 = -.512$

(x^2-1) changes sign across $x = \pm 1$ (neg inside, positive outside)

$(5x^2-1)$ changes sign across $x = \pm 1/\sqrt{5}$ (neg inside, positive outside)



so 4 points of inflection

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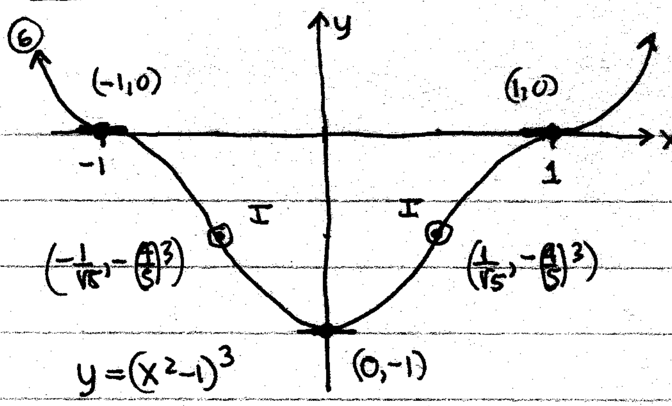
⑥



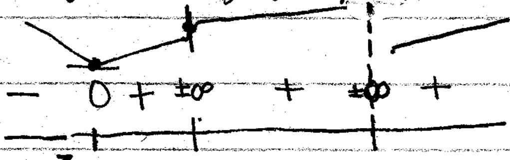
⑥ e) $x=0$: $y=f(0)=\frac{0}{1}=0$; $0=y=\frac{x^{1/3}}{1-x} \to x^{1/3}=0 \to x=0$ } (0,0) only intercept

f) $f'(x)=\dfrac{2x+1}{3x^{2/3}(1-x)^2}$ $=0 \to 2x+1=0 \to x=-1/2$

$y=f(-\tfrac12)=\dfrac{(-1/2)^{1/3}}{1-(-1/2)}=\dfrac{2}{3\sqrt[3]{2}} \approx -.53$

$\hookrightarrow =0$ $f' \to \pm\infty$ also critical pt.

sgn $f'(x)$ = sgn $(2x+1)$ → since denom squares



local min → $-\frac12$

↑ vertical tangent.

⑦ a) $f(x)=\dfrac{x^{1/3}}{1-x}$

$f'(x)=\dfrac{(1-x)\,\frac13 x^{-2/3}-x^{1/3}(-1)}{(1-x)^2}=\dfrac{\frac{1-x}{3x^{2/3}}+\frac{x^{1/3}}{1}}{(1-x)^2}$

$=\dfrac{\frac{(1-x)+3x^{2/3}x^{1/3}}{3x^{2/3}}}{(1-x)^2}=\dfrac{1-x+3x}{3x^{2/3}(1-x)^2}=\boxed{\dfrac{2x+1}{3x^{2/3}(1-x)^2}}=u$

g) $f''(x)=\dfrac{2(5x^2+5x-1)}{9x^{5/3}(1-x)^3}=0 \to 5x^2+5x-1=0$

$\hookrightarrow =0.$

$f'' \to \pm\infty$ & changes sign pt of inflection.

$x=\dfrac{-5\pm\sqrt{5^2-4(5)(-1)}}{2\cdot 5}=\dfrac{-5\pm\sqrt{45}}{10}=-\frac12\pm\frac{3\sqrt5}{10}$

$\approx -1.17, .17$

$y=f(x_\pm)$

$\approx -.49, .67$

b)

$f''(x)=\dfrac13\dfrac{\left[x^{2/3}(1-x)^2(2)-(2x+1)\left(\frac23 x^{-1/3}(1-x)^2+x^{2/3}(2)(1-x)(-1)\right)\right]}{x^{4/3}(1-x)^4}$

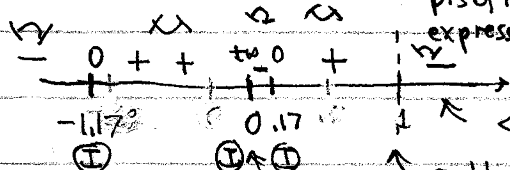
$=\dfrac{2(1-x)}{3x^{4/3}(1-x)^4}\left[x^{2/3}(1-x)-(2x+1)\left(\frac{(1-x)}{3x^{1/3}}-\frac{x^{2/3}}{1}\right)\right]$

$=\dfrac{2}{3x^{4/3}(1-x)^3}\left[\dfrac{3x^{1/3}x^{2/3}(1-x)-(2x+1)(1-x-3x^{1/3}x^{2/3})}{3x^{1/3}}\right]$

$=\dfrac{2}{9x^{5/3}(1-x)^3}\underbrace{\left[3x(1-x)-(2x+1)(1-\overbrace{x-3x}^{-4x})\right]}_{\underbrace{3x-3x^2-2x-1+8x^2+4x}_{5x^2+5x-1}}$

$=\boxed{\dfrac{2(5x^2+5x-1)}{9x^{5/3}(1-x)^3}}$

pts of inflection — quadratic expression changes sign.



<0 when x=2.

switches sign across 1−x

switches sign across 0

j) increasing: $(-\tfrac12,0),(0,1),(1,\infty)$

decreasing: $(-\infty,-\tfrac12)$

concave up: $(-1.17,0)$, $(.17,1)$

concave down: $(-\infty,-1.17)$, $(0,.17)$, $(1,\infty)$

c) $\ln u=\ln(2x+1)-\frac23\ln x-2\ln|1-x|-\ln 3$

$\dfrac1u\dfrac{du}{dx}=\dfrac{2}{2x+1}-\dfrac{2}{3x}-\dfrac{2(-1)}{1-x}=2\left[\dfrac{3x(1-x)}{(2x+1)3x(1-x)}-\dfrac{(2x+1)(1-x)}{3x(2x+1)(1-x)}+\dfrac{(2x+1)(3x)}{(1-x)(2x+1)(3x)}\right]$

$=2\left[\dfrac{3x-3x^2-(2x+1-2x^2-x)+6x^2+3x}{3x(2x+1)(1-x)}\right]=\dfrac{2(5x^2+5x-1)}{3x(2x+1)(1-x)}$

$f''(x)=\dfrac{du}{dx}=u\,\dfrac{2(5x^2+5x-1)}{3x(2x+1)(1-x)}=\dfrac{(2x+1)}{3x^{2/3}(1-x)^2}\,\dfrac{2(5x^2+5x-1)}{3x(2x+1)(1-x)}$

$=\dfrac{2(5x^2+5x-1)}{9x^{5/3}(1-x)^3}$ ✓

d) see MAPLE work

e) domain: $x\neq 1$ division by zero $\frac10\sim\infty$ Vertical asymptote: $x=1$

no symmetry

$\lim\limits_{x\to\pm\infty}\dfrac{x^{1/3}}{1-x}=\lim\limits_{x\to\pm\infty}\dfrac{x^{1/3}/x}{\frac1x-1}=\lim\limits_{x\to\pm\infty}\dfrac{1}{\underbrace{x^{2/3}}_{>0}(\underbrace{\tfrac1x}_{\to 0}-1)}=0$, horizontal asymptote: $y=0$