9.1

1

curvature ?

First the tensor, then the interpretation

$\mathbb{R}^n$, cartesian coords: $\{x^i\}$

$\nabla_i Z^k = \partial_i Z^k$

$$\nabla_j \nabla_i Z^k = \partial_j \partial_i Z^k = \partial_i \partial_j Z^k = \nabla_i \nabla_j Z^k$$

$$(\nabla_j \nabla_i - \nabla_i \nabla_j) Z^k = 0$$

$$[\nabla_j, \nabla_i] Z^k = 0 \qquad Z^k{}_{;ij} - Z^k{}_{;ji} = 2 Z^k{}_{;[ij]} = 0$$

commutator of covariant derivatives vanishes

what about

$$[\nabla_X, \nabla_Y] Z^k = \underbrace{[X^i \nabla_i, Y^j \nabla_j]}\, Z^k \,?$$

$$\left.\begin{aligned} &= [X^i \partial_i, Y^j \partial_j] Z^k \\ &= [X, Y] Z^k \\ &= \nabla_{[X,Y]} Z^k \end{aligned}\right\} \text{ in cartesian coords.}$$

$$\therefore \underbrace{\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)}_{\text{coordinate independent !}} Z^k = 0$$

$$\nabla_Y Z^i = Z^i{}_{;j} Y^j = Z^i{}_{,j} Y^j \qquad \text{cartesian coords}$$

check in detail

$$[\nabla_X \nabla_Y - \nabla_Y \nabla_X] Z^i = [\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z)]^i$$

$$= (\nabla_Y Z)^i{}_{,k} X^k - (\nabla_X Z)^i{}_{,k} Y^k$$

$$= (Z^i{}_{,j} Y^j)_{,k} X^k - (Z^i{}_{,j} X^j)_{,k} Y^k$$

$$= Z^i{}_{,jk} Y^j X^k - Z^i{}_{,jk} X^j Y^k \qquad \text{switch dummy indices}$$

$$+ Z^i{}_{,j} Y^j{}_{,k} X^k - Z^i{}_{,j} X^j{}_{,k} Y^k$$

$$= \underbrace{[Z^i{}_{,jk} - Z^i{}_{,kj}]}_{=0} X^k Y^j + \underbrace{Z^i{}_{,j} \underbrace{(X^k Y^j{}_{,k} - Y^k X^j{}_{,k})}_{[X,Y]^j}}_{\nabla_{[X,Y]} Z^i}$$

$$\therefore (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z^i = 0$$

$\boxed{9.1}$

1b

HISTORICALLY reverse calculation

$$\nabla_i X^k = \partial_i X^k + \Gamma^k{}_{im} X^m$$

$$\partial_i X^k = \nabla_i X^k - \Gamma^k{}_{im} X^m$$

$$\partial_j \partial_i X^k = \nabla_j (\nabla_i X^k - \Gamma^k{}_{im} X^m) - \Gamma^k{}_{jn} (\nabla_j X^n - \Gamma^n{}_{im} X^m)$$

$$\partial_i \partial_j X^k = \dots$$

$$0 = (\partial_j \partial_i - \partial_i \partial_j) X^k = \dots$$

integrability condition for PDEs for cov constant vector field

Now evaluate in general coords:

$$\nabla_Y Z^i = Z^i{}_{;j} Y^j = (Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m) Y^j$$

$$[\nabla_X \nabla_Y Z]^i = ([\nabla_Y Z]^i{}_{,k} + \Gamma^i{}_{km} [\nabla_Y Z]^m) X^k$$

$$= \{\underbrace{[(Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m) Y^j]_{,k}}_{\text{expand}} + \underbrace{\Gamma^i{}_{km} (Z^m{}_{,j} + \Gamma^m{}_{jp} Z^p) Y^j \} X^k}_{\text{no change}}$$

$$= \{(\underset{①}{Z^i{}_{,jk}} + \underset{②}{\Gamma^i{}_{jm} Z^m{}_{,k}} + \underset{③}{\Gamma^i{}_{jm,k}}) Y^j + \boxed{\underbrace{(Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m)}_{④\ Z^i{}_{;j}} Y^j{}_{,k}}$$

$$+ \Gamma^i{}_{km} (\underset{⑤}{Z^m{}_{,j}} + \underset{⑥}{\Gamma^m{}_{jp} Z^p}) Y^j \} X^k$$

$$= [\underset{①}{Z^i{}_{,jk}} + \underset{②}{\Gamma^i{}_{jm} Z^m{}_{,k}} + \underset{⑤}{\Gamma^i{}_{km} Z^m{}_{,j}} + \underset{③}{\Gamma^i{}_{jm,k}} + \underset{⑥}{\Gamma^i{}_{km} \Gamma^m{}_{jp} Z^p}] X^k Y^j$$

$$+ \underset{④}{Z^i{}_{;j} Y^j{}_{,k} X^k}$$

if we switch X, Y and subtract → antisymmetrize

symmetric terms in (jk) cancel out.

$$①,\quad ②+⑤ \longrightarrow 0$$

upper line contribution

$$[\nabla_X \nabla_Y Z]^i = [\ldots]^i{}_{jk} X^k Y^j \quad \text{relable}$$

$$-[\nabla_Y \nabla_X Z]^i = -[\ldots]^i{}_{jk} X^j Y^k = -[\ ]^i{}_{kj} X^k Y^j$$

$$[\nabla_X \nabla_Y - \nabla_Y \nabla_X] Z^i = ([\ldots]^i{}_{jk} - [\ldots]^i{}_{kj}) X^k Y^j$$

← symmetric terms cancel

only ③+⑥ remain in difference — antisym in (j,k)

lower line contribution

$$Z^i{}_{;j} \underbrace{(X^k Y^j{}_{,k} - Y^k X^j{}_{,k})}_{[X,Y]^j}$$

$$(\nabla_{[X,Y]} Z)^i$$

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z^i$$

$$= (\Gamma^i{}_{jm,k} - \Gamma^i{}_{km,j} + \Gamma^i{}_{km} \Gamma^m{}_{jp} - \Gamma^i{}_{jm} \Gamma^m{}_{kp}) \underbrace{X^k Y^j Z^m}_{\text{no derivatives !!}}$$

tensor!! $R^i{}_{mkj}$ ← multilinear function

$\boxed{9.1}$
3

$\mathbb{R}^n$, any coords, $R^i{}_{jk\ell} \equiv 0$

$$R^i{}_{jmn} = \underbrace{\partial_m \Gamma^i{}_{nj} - \partial_n \Gamma^i{}_{mj}}_{\text{2nd derivatives of } g_{ij}} + \Gamma^i{}_{mk}\Gamma^k{}_{nj} - \Gamma^i{}_{nk}\Gamma^k{}_{mj}$$

i, j: linear trans; mn: 2-form

$+$ first derivatives of g_{ij}

this is a nightmare BUT zero for the flat space flat connection

Yes, this is our measure of curvature.

Remember calc I: $\frac{d^2}{dx^2} f(x) \to$ concavity $\to$ curvature

second derivatives

smells right.

later we interpret this in terms of parallel transport.

$$R = R^i{}_{jmn} \underbrace{e_i \otimes \omega^j}_{\text{linear transformation}} \otimes \omega^m \otimes \omega^n$$

only antisymmetric part contributes

$$= \frac{1}{2} R^i{}_{jmn}\, e_i \otimes \omega^j \otimes \underbrace{(\omega^m \wedge \omega^n)}_{\text{2-form}}$$

partial evaluation on lower 3 indices $\to$ $R(X,Y)Z = \frac{1}{2} R^i{}_{jmn}\, e_i\, \omega^j(Z)\, \omega^m(X)\, \omega^n(Y)$

(X,Y): evaluate 2-form factor $\to$ picks out plane of $X \wedge Y$

Z: undergoes linear transformation

"linear transformation valued 2-form"

measures curvature in any space with metric g_{ij} giving rise to metric connection ∇ with components $\Gamma^i{}_{jk}$ in any coord system

$\boxed{9.1}$
4

Frames often more useful for interpretation than coordinates

Redo calculation in general frame $\{e_i\}$, dual frame $\{\omega^i\}$

$$R^i{}_{jmn} = R(\omega^i, \underset{Z}{e_j}, \underset{X,\,Y}{e_m, e_n}) \downarrow$$

definition of frame components

$$(\nabla_{e_m}\nabla_{e_n} - \nabla_{e_n}\nabla_{e_m} - \nabla_{[e_m,e_n]})\, e_j = R^i{}_{jmn}\, e_i$$

$$\nabla_{e_n} e_j = \Gamma^k{}_{nj}\, e_k$$

$$\nabla_{e_m}\nabla_{e_n} e_j = \nabla_{e_m}(\Gamma^k{}_{nj} e_k) = \underbrace{(\nabla_{e_m}\Gamma^k{}_{nj})}_{\Gamma^k{}_{nj,m}} e_k + \Gamma^k{}_{nj} \underbrace{\nabla_{e_m} e_k}_{\Gamma^\ell{}_{mk} e_\ell}$$

switch & subtract

switch

$$\Gamma^\ell{}_{nj}\,\Gamma^k{}_{m\ell}\, e_k$$

$$= (\Gamma^k{}_{nj,m} + \Gamma^k{}_{m\ell}\Gamma^\ell{}_{nj})\, e_k$$

$$(\nabla_{e_m}\nabla_{e_n} - \nabla_{e_n}\nabla_{e_m})\, e_j = \begin{pmatrix} \Gamma^k{}_{nj,m} + \Gamma^k{}_{m\ell}\Gamma^\ell{}_{nj} \\ - \Gamma^k{}_{mj,n} - \Gamma^k{}_{m\ell}\Gamma^\ell{}_{mj} \end{pmatrix} e_k$$

$$-\nabla_{[e_m,e_n]}\, e_j = -(\nabla_{C^k{}_{mn} e_k})\, e_j = -C^k{}_{mn} \underbrace{\nabla_{e_k} e_j}_{\Gamma^\ell{}_{kj} e_\ell}$$

switch k,ℓ

$$R^k{}_{jmn}\, e_k = (\Gamma^k{}_{nj,m} - \Gamma^k{}_{mj,n} + \Gamma^k{}_{m\ell}\Gamma^\ell{}_{nj} - \Gamma^k{}_{n\ell}\Gamma^\ell{}_{mj} - C^\ell{}_{mn}\Gamma^k{}_{\ell j})\, e_k$$

just one extra term
zero for coordinate frame
$[\partial_i, \partial_j] = 0$

$\boxed{9.1}$

5

$R_{ijmn} = g_{ik} R^k{}_{jmn}$ fully covariant curvature tensor

antisymmetric in 2nd pair of indices $= -R_{ijnm}$
(2-form definition)

antisymmetric in first pair of indices
(parallel transport leads to rotation → generator of rotations
antisymmetric matrix
when index lowered.

$= -R_{jimn}$

$= R_{mnij}$ → symmetric in pair interchange
(like a bilinear function on 2-vectors)

$R_{mnij} S^{mn} T^{ij} = R_{mnij} T^{mn} S^{ij}$

how to prove? — manipulate formula with index lowered (see text)

how to prove? $g_{ij;m} = 0 \rightarrow g_{ij;mn} = 0$

$\rightarrow g_{ij;mn} - g_{ij;nm} = 0$

$[\nabla_m, \nabla_n] g_{ij} = 0$

$\downarrow \cdots$

$R_{[mn]ij} = 0$

$\boxed{9.1}$

6

another symmetry

recall Jacobi identity ← $[X,Y]=XY-YX$

$$\underbrace{[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]}_{\text{cyclic permute}} = \underbrace{\ldots(6\text{ terms})}_{\text{cancel in pairs.}} = 0$$

recall symmetry of connection

$\Gamma^i{}_{jk}=\Gamma^i{}_{kj}$ in coords

$$\nabla_X Y-\nabla_Y X = \ldots = [X,Y]$$ connection components cancel.

$$\nabla_Z X-\nabla_X Z=[Z,X]$$

$X \to [X,Y]$

$$\nabla_Z \underbrace{[X,Y]}_{\nabla_X Y-\nabla_Y X}-\nabla_{[X,Y]}Z = [Z,[X,Y]]$$

$$\nabla_Z\nabla_X Y-\nabla_Z\nabla_Y X - \nabla_{[X,Y]}Z = [Z,[X,Y]]$$

$$\nabla_X\nabla_Y Z-\nabla_X\nabla_Z Y - \nabla_{[Y,Z]}X = [X,[Y,Z]]$$

$$\nabla_Y\nabla_Z X-\nabla_Z\nabla_Y X - \nabla_{[Z,X]}Y = [Y,[Z,X]]$$

cyclic permute

sum

$$R(Z,X)Y + R(X,Y)Z + R(Y,Z)X = 0$$ Jacobi Identity

$$\underbrace{R^m{}_{ijk}+R^m{}_{jki}+R^m{}_{kij}}_{\text{cyclic premute}} = 0$$ ← "Blanchi Identity" (consequence of symmetric connection)

$$= \frac{1}{2}\left(R^m{}_{ijk}+R^m{}_{jki}+R^m{}_{kij} - R^m{}_{ikj}-R^m{}_{jik}-R^m{}_{kji}\right)$$

$$= \frac{1}{2}\,3!\,R^m{}_{\underbrace{[ijk]}_{\text{antisymmetric part}}}$$

$\boxed{9.1}$

7

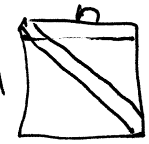
how many independent components? $\underline{n \text{ dimensions}}$

$R_{ij\,mn}$

$N = \frac{n(n-1)}{2}$

$N \times N$ sym matrix

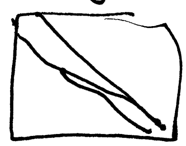
2-form = antisym matrix



n

$n-1$
$+\, n-2$
$+ \cdots$
$+\, 1$
$+\, 0$

$\frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$



N

N
$N-1$
$\vdots$
$+1$

$\frac{N(N+1)}{2}$

symmetric linear function of bi-vectors

$$\frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}+1\right)}{2} = \frac{1}{8}\, n(n-1)(n^2-n+2)$$

$R^m{}_{ijk} = 0$ ← how many conditions?
must subtract this number
at midnight I could not figure this out

Final number (google)

n	$\frac{1}{12} n^2(n^2-1)$	without Bianchi
1	0	0
2	1	
3	6	6
4	20	21 ← 1 extra condition

} Bianchi nothing extra. consequence of other symmetries

Exercise: find explanation of this number

$R^m{}_{\cdot ijk} \to R^m{}_{imk} \equiv R_{ik} = R_{ki}$ symmetric Ricci

$R^{mi}{}_{mi} = R^i{}_i \equiv R$ scalar curvature

$G_{ij} \equiv R_{ij} - \frac{1}{2} R\, g_{ij}$ Einstein tensor

$\boxed{9.1}$

8

$\boxed{n=3:}$

$0 = 3R_{1[123]} = \underbrace{\cancel{R_{1123}}}_{0} + R_{1231} + \underbrace{R_{1312}}_{R_{1213}} \rightarrow R_{12[31]} = 0$ not new

$\cancel{R_{1231}}$

$\boxed{n=4:}$

$0 = 3R_{4[123]} = R_{4123} + R_{4231} + R_{4312}$

$0 = 3R_{4[423]} = \cancel{R_{4423}} + R_{4234} + \underbrace{R_{4342}}_{R_{4243}} \Rightarrow R_{42[34]} = 0$ not new

if repeated

all ~~others have at least one repeated~~.

$0 = 3R_{1[234]} = R_{1234} + R_{1342} + R_{1423}$

$R_{3412} + R_{4213} - R_{4123}$

$- R_{4312} - R_{4231} - R_{4123}$ same as first.

four choices for index leftout.

all collapse to 1 independent constraint.

$\boxed{n=2}$ $R_{2[\underbrace{122}]}$

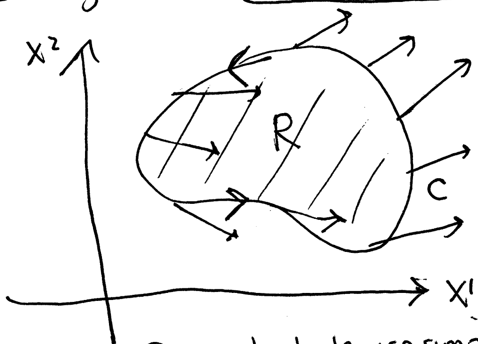
3-forms identically zero

how to analyze in general?

$\boxed{9.2}$ limitingly small loop parallel transport and curvature

1

Background : $\boxed{\text{Green's theorem in plane}}$



$\mathbb{X} = \langle X_1, X_2 \rangle$

$$\oint_C \mathbb{X}_1 dx^1 + \mathbb{X}_2 dx^2 \qquad \leftarrow \text{integral of } \underline{\underline{1\text{-form}}}$$

$$= \iint_R \underbrace{\left(\frac{\partial \mathbb{X}_2}{\partial x^1} - \frac{\partial \mathbb{X}_1}{\partial x^2} \right)}_{\text{curl} \langle X_1, X_2, 0 \rangle \cdot \hat{k}} dA$$

3rd component of curl as 3-component field.

C: counterclockwise simple closed curve

$C(\lambda)$ parametrized curve $\quad C(0) = C(a)$ initial & final pt of loop

Why?

Define $Z(\lambda)$ by $\begin{cases} \dfrac{dZ(\lambda)}{d\lambda} = \mathbb{X}_i(C(\lambda)) \dfrac{dx^i(\lambda)}{d\lambda} \\ Z(0) = Z_0 \end{cases}$ $\quad$ <u>IVP</u> has unique solution

solution :

$$\int_0^a \frac{dZ}{d\lambda} d\lambda = \int_0^a \mathbb{X}_i \frac{dx^i}{d\lambda} d\lambda = \oint_C \mathbb{X}_i dx^i$$

$$\begin{aligned} &\shortparallel \\ &Z(a) - Z(0) \\ &\shortparallel \\ &Z(a) - Z_0 = \Delta Z \quad \text{increment} = \text{line integral of } \mathbb{X} \end{aligned}$$

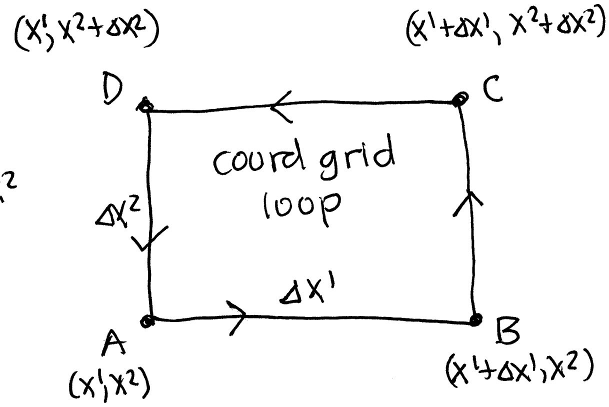
so what? no, this allows us to better understand the proof.

$\boxed{9.2}$

2

$$\boxed{\begin{array}{l}\Delta x^1 \to 0 \\ \Delta x^2 \to 0\end{array}}$$ shrink loop to point

$$-\int_{x^1}^{x^1+\Delta x^1} X_1(\underset{\lambda}{x^1}, x^2+\Delta x^2)\, d\underset{\lambda}{x^1} \approx -X_1(x^1, x^2+\Delta x^2)\,\Delta x^1$$



$$-\int_{x^2}^{x^2+\Delta x^2} X_2(x^1, \underset{\lambda}{x^2})\, d\underset{\lambda}{x^2}$$

$$\approx -X_2(x^1, x^2)\,\Delta x^2$$

$$\int_{x^2}^{x^2+\Delta x^2} X_2(x^1+\Delta x^1, \underset{\lambda}{x^2})\, \underset{\lambda}{dx^2}$$

$$\approx X_2(x^1+\Delta x^1, x^2)\,\Delta x^2$$

$$\int_{x^1}^{x^1+\Delta x^1} X_1(x^1, x^2)\, dx^1$$ dummy variable

$$\int_{x^1}^{x^1+\Delta x^1} X_1(\lambda, x^2)\, d\lambda$$

$$\approx X_1(x^1, x^2)\,\Delta x^1$$

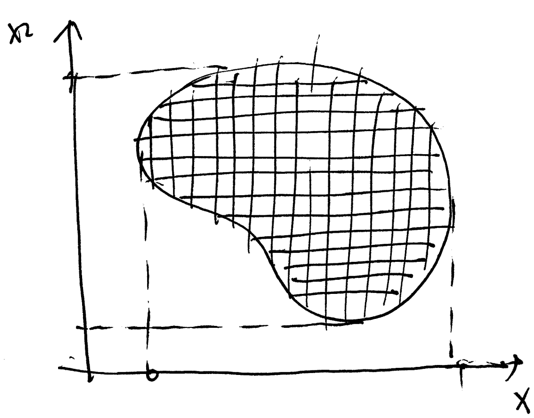
$\Delta Z =$

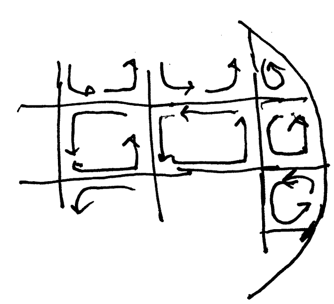
$$\oint_C X_i\,dx^i \approx \underbrace{[X_1(x^1,x^2) - X_1(x^1, x^2+\Delta x^2)]}_{\approx\, -\frac{\partial X_1}{\partial x^2}(x^1,x^2)\,\Delta x^2}\Delta x^1 + \underbrace{[X_2(x^1+\Delta x^1, x^2) - X_2(x^1,x^2)]}_{\approx\, \frac{\partial X_2}{\partial x^1}(x^1,x^2)\,\Delta x^1}\Delta x^2$$

$$\approx \left[\frac{\partial X_2}{\partial x^1}(x^1,x^2) - \frac{\partial X_1}{\partial x^2}(x^1,x^2)\right]\Delta x^1 \Delta x^2$$

$$\approx \iint_R \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right) dA$$

Green's thm true in limit $\Delta A \to 0$





internal grid contributions all cancel
only boundary curve contributions survive

$$\iint \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right) dA$$

$$= \lim_{n\to\infty} \sum_{i,j=1}^{n} \left(\frac{\partial X_2}{\partial x^1} - \frac{\partial X_1}{\partial x^2}\right)\Delta A = \ldots = \oint_C X_i\,dx^i$$

9.2
3

Parallel transport: $k = 1, \ldots, n$

$$\frac{DZ^k}{d\lambda} = \frac{dZ^k}{d\lambda} + \Gamma^k{}_{ij} Z^j \frac{dx^i}{d\lambda} = 0 \rightarrow \frac{dZ^k}{d\lambda} = \underbrace{-\Gamma^k{}_{ij} Z^j}_{\equiv \Xi^k_i} \frac{dx^i}{d\lambda}$$

$Z^k(0) \equiv Z^k_0$

IVP unique soln for each k

just k scalars instead of one: $Z \to Z^k$

focus on x^1, x^2 coord grid box

$$\Delta Z \approx (\Xi_{2,1} - \Xi_{1,2}) \Delta x^1 \Delta x^2 \longrightarrow \Delta Z^k \approx (\Xi^k_{2,1} - \Xi^k_{1,2}) \Delta x^1 \Delta x^2$$

$$= \left[-(\Gamma^k{}_{2i} Z^i)_{,1} + (\Gamma^k{}_{1i} Z^i)_{,2}\right] \Delta x^1 \Delta x^2$$

$$= (-\Gamma^k{}_{2i,1} Z^i - \Gamma^k{}_{2i} \text{"}Z^i{}_{,1}\text{"} + \Gamma^k{}_{1i,2} Z^i + \Gamma^k{}_{1i} \text{"}Z^i{}_{,2}\text{"}) \Delta x^1 \Delta x^2$$

$$\text{"}\frac{dZ^i}{dx^1}\text{"} = -\Gamma^i{}_{1j} Z^j \underbrace{\frac{dx^1}{d\lambda}}_{=1}$$

$$\text{"}\frac{dZ^i}{dx^2}\text{"} = -\Gamma^i{}_{2j} Z^j \frac{dx^2}{d\lambda} = 1$$

derivative along coord line $x^i = \lambda$

$$\frac{\Delta Z^k}{\Delta x^1 \Delta x^2} \approx -\Gamma^k{}_{2i,1} Z^i + \Gamma^k{}_{2i} \Gamma^i{}_{1j} Z^j + \Gamma^k{}_{1i,2} Z^i - \Gamma^k{}_{1i} \Gamma^i{}_{2j} Z^j$$

swap dummy indices

$$= -\Gamma^k{}_{2i,1} Z^i + \Gamma^k{}_{2j} \Gamma^j{}_{1i} Z^i + \Gamma^k{}_{1i,2} Z^i - \Gamma^k{}_{1j} \Gamma^j{}_{2i} Z^i$$

$$= -(\Gamma^k{}_{2i,1} - \Gamma^k{}_{1i,2} + \Gamma^k{}_{2j} \Gamma^j{}_{1i} - \Gamma^k{}_{2i} \Gamma^i{}_{2j}) Z^i$$

$$= -R^k{}_{i12} Z^i$$

$$\boxed{\Delta Z^k \approx -R^k{}_{i12} Z^i \Delta x^1 \Delta x^2}$$

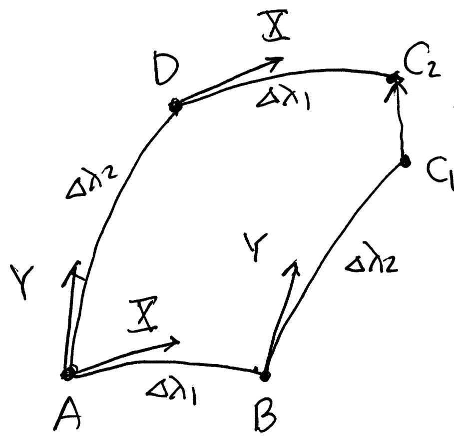
curvature is measured in 2-plane cross-sections of tangent spaces

mn for each x^m-x^n coordinate rectangle

$$= -\underbrace{\Omega^k{}_i(\partial_1, \partial_2)}_{\Omega_{(ki)} = 0 \text{ rotation generator}} Z^i$$

$\Omega^k{}_i = \frac{1}{2} R^k{}_{imn}\, dx^m \wedge dx^n$

curvature 2-form matrix



$\longleftarrow -[X,Y]\,\Delta\lambda_1\Delta\lambda_2$ $\left\{\begin{array}{l}\text{extra contribution}\\ \text{to loop}\\ \text{calculation}\end{array}\right.$

move along flow lines of 2 noncommuting vector fields

$[X,Y] \to 0$ reduces discussion to previous one of coordinate grid boxes

frame: $\{e_i\}$

$$(X,Y) \to (e_1,e_2) \to [e_1,e_2] = C^k{}_{12}\, e_k$$

$$-[X,Y]\,\Delta\lambda_1\Delta\lambda_2 = \underbrace{-C^i{}_{12}\, e_i\, \Delta\lambda_1\Delta\lambda_2}_{\text{tangent to curve segment}}$$

reverses sign

$$\Delta Z^k \approx \cdots + \underbrace{\Gamma^k{}_{ij} Z^j \frac{dx^i}{d\lambda}} + \Gamma^k{}_{ij}\, C^i{}_{12}\, Z^j\, \Delta\lambda_1\Delta\lambda_2$$

e_1, e_2 loop

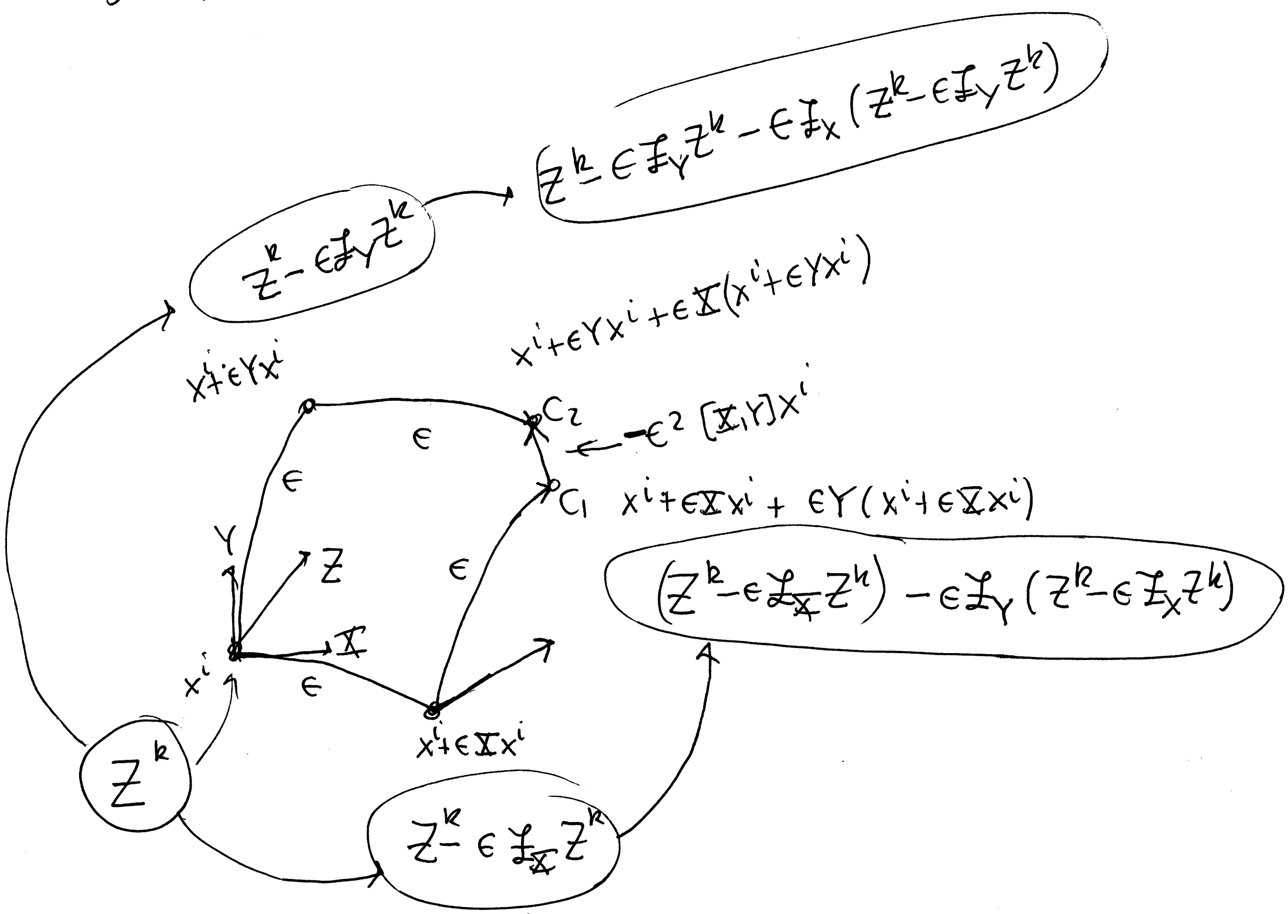
$$\approx \Big[-\big(\underbrace{R^k{}_{j12}}_{\text{coord formula}} - \underbrace{\Gamma^k{}_{ij} C^i{}_{12}}\big) Z^j \Big]\, \Delta\lambda_1\Delta\lambda_2$$

extra term in formula corresponding to this extra contribution

$\boxed{9.2}$
5

instead of parallel transport

closing quadrilateral flowing along flow lines & dragging along a vector field Z



extra closer segment $\quad Z^k_{C_2} - Z^k_{C_1} \approx -\mathcal{L}_{\epsilon^2 [X,Y]} Z^k$

$\wr$

$\epsilon^2 [\mathcal{L}_X, \mathcal{L}_Y] Z$

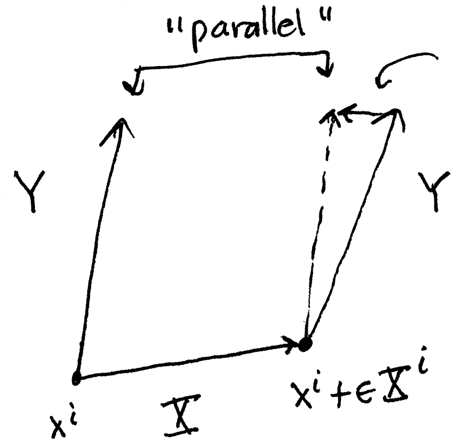
check sign?

$$\boxed{[\mathcal{L}_X, \mathcal{L}_Y] Z = -\mathcal{L}_{[X,Y]} Z}$$

$$\left([\mathcal{L}_X, \mathcal{L}_Y] - \mathcal{L}_{[X,Y]}\right) Z = 0$$

compare: $\updownarrow$

$$\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) Z = R(X,Y) Z$$

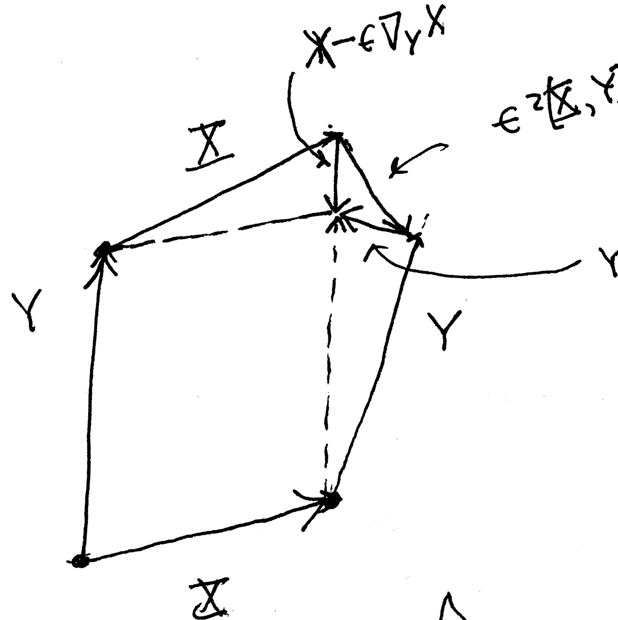


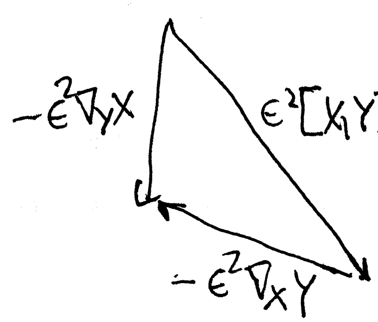
$Y - \epsilon \nabla_X Y$

subtract off cov derivative increment to get parallel transported vector

$\underbrace{}_{}$ increment $\lambda = \epsilon$ along flowline of X

"closer of quadrilateral" of flowlines of X and Y





$$\epsilon^2 [X, Y] = -\epsilon^2 \nabla_Y X + \epsilon^2 \nabla_X Y$$

$$\underbrace{\nabla_X Y - \nabla_Y X}_{} = [X, Y]$$

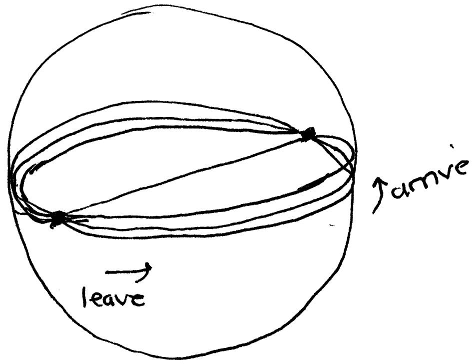
"closes" quadrilateral of X & Y

If symmetric connection.

incorporated into curvature as Bianchi identity

$\boxed{9.4}$ 1

geometry as geodesic lensing

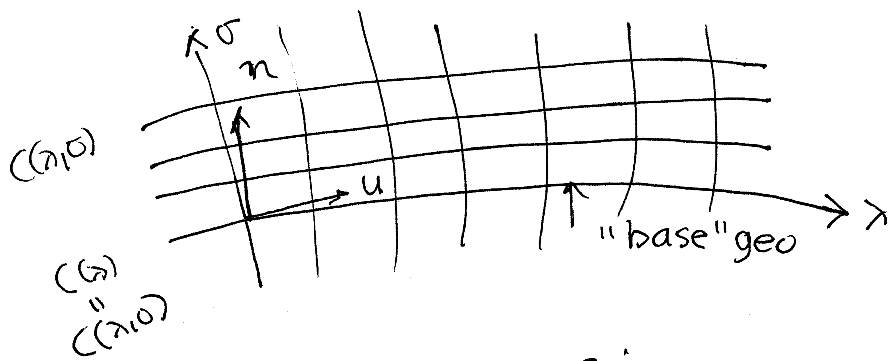


tilt great circles about a common diameter from the equator

get family of nearby geodesics which start out diverging but are forced to converge by the positive curvature.

how to generalize to any surface / any space?

1-parameter family of affinely parametrized geos = 2-surface



$$c^i(\lambda) = x^i(C(\lambda))$$
$$c^i(\lambda,\sigma) = x^i(C(\lambda,\sigma))$$

- 2 tangents:

$$u^i(\lambda,\sigma) = \frac{\partial c^i}{\partial \lambda}(\lambda,\sigma)$$
$$n(\lambda,\sigma) = \frac{\partial c^i}{\partial \sigma}(\lambda,\sigma)$$

$\downarrow \sigma = 0$

$u^i(\lambda)$, $n^i(\lambda)$ along "base" geo

$$\frac{Du}{d\lambda} = 0 \qquad \hookrightarrow u^i = \frac{dx^i}{d\lambda} = \frac{Dx^i}{d\lambda} \rightarrow \frac{D^2x^i}{d\lambda^2} = 0$$

- symmetry of metric connection

$$\nabla_u n - \nabla_n u = [u,n] \leftarrow [u,n] = 0$$

form coord grid on surface.

$$= 0$$

$\downarrow$ derivatives along gridlines of surface $\left\{ \begin{array}{l} \nabla_u \rightarrow \frac{D}{d\lambda} \\ \nabla_n \rightarrow \frac{D}{d\sigma} \end{array} \right.$ when acting on tensors defined on 2-surface

$$\frac{Dn}{d\lambda} - \frac{Du}{d\sigma} = 0$$

definition of curvature:

$$(\nabla_u \nabla_\eta - \nabla_\eta \nabla_u) u^i = R^i{}_{jmn} u^j u^m \eta^n$$

$$\downarrow \; \|$$

$$\frac{D}{d\lambda}\left(\frac{Du^i}{d\sigma}\right) - \frac{D}{d\sigma}\left(\underbrace{\frac{Du^i}{d\lambda}}_{=0 \text{ geo}}\right)$$

$\frac{D\eta^i}{d\lambda}$ ← sym connection

$$\frac{D^2\eta^i}{d\lambda^2} = R^i{}_{jmn} u^j u^m \eta^n = -R^i{}_{j\underbrace{nm}_{\text{central}}} u^j u^m \eta^n$$

$$\boxed{\frac{D^2\eta^i}{d\lambda^2} + R^i{}_{jmn} u^j \eta^m u^n = 0}$$

Jacobi geodesic deviation eqn

sols = Jacobi fields

$\downarrow$ $\lambda = s$ arclength parametrization $\quad u = \hat{u} \equiv \hat{T}$ unit tangent

$$\boxed{\frac{D^2\eta^i}{ds^2} + R^i{}_{jmn} T^j \eta^m T^n = 0}$$

project η perpendicular to $\hat{u} = \hat{T}$: $\quad \eta = \rho \hat{N} + \alpha \hat{T} \qquad \hat{N} \cdot \hat{T} = 0$

choose $\frac{D\hat{N}}{ds} = 0$ (?) $\left(= \frac{D\hat{T}}{ds}\right)$ both covariant constant along C

1st term

$$\frac{D^2}{ds^2}(\rho \hat{N} + \alpha \hat{T}) = \left(\frac{d^2\rho}{ds^2}\right)\hat{N} + \left(\frac{d^2\alpha}{ds^2}\right)\hat{T}$$

2nd term

$$R^i{}_{jmn} T^j (\rho N^m + \alpha T^m) T^n = \underbrace{\rho R^i{}_{jmn} T^j N^m T^n} + \underbrace{\alpha R^i{}_{jmn} N^j T^m T^n}_{=0}$$

$$T_i R^i{}_{jmn} T^j N^m T^n = R_{ijmn} T^i T^j N^m T^n = 0 \quad \text{orthogonal to } T$$

$$= \rho (N_k R^k{}_{jmn} T^j N^m T^n) \frac{N^i}{N_\ell N^\ell} = \rho \, \mathrm{sgn}(\hat{N}) (R_{jkmn} N^j T^k N^m T^n) N^i$$

→ $\hat{T}$ component: $\frac{d^2\alpha}{ds^2} \neq 0$

→ $\hat{N}$ component: $\boxed{\frac{d^2\rho}{ds^2} + K\rho = 0}$, $\quad K = R^{\hat{N}}{}_{\hat{T}\hat{N}\hat{T}} = \mathrm{sgn}(\hat{N}) R_{\hat{N}\hat{T}\hat{N}\hat{T}}$

"limiting" geo separation distance

Gaussian curvature (modulo sign?)

$\boxed{9.4}$
3

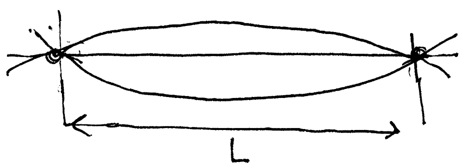
solutions for K constant along geo:

$K>0$: $\dfrac{d^2 y}{ds^2} + \underbrace{K}_{\omega^2} y = 0 \rightarrow y = c_1 \cos \omega s + c_2 \sin \omega s$
$= c_1 \cos\left(\frac{2\pi s}{\lambda}\right) + c_2 \sin\left(\frac{2\pi s}{\lambda}\right)$

· frequency $\omega > 0$
wavelength $\lambda = \frac{2\pi}{\omega}$

$L = \frac{\lambda}{2}$ halfwavelength = arclength between zeros of y

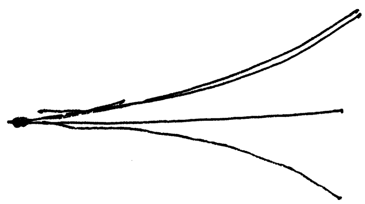
"stable"



geos reconverge at a point

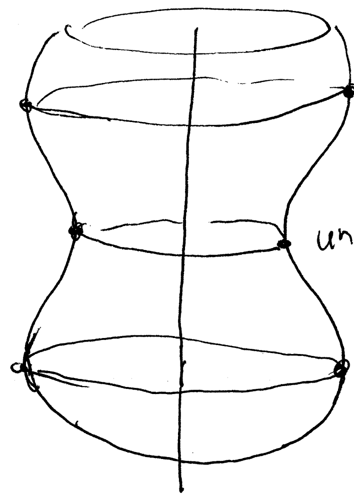
$K<0$: $\dfrac{d^2 y}{ds^2} + \underbrace{K}_{-k^2} y = 0 \rightarrow y = c_1 \cosh ks + c_2 \sinh ks$
$= \underbrace{c_1 e^{ks}}_{\text{growth}} + \underbrace{c_2 e^{-ks}}_{\text{decay}}$ (as s increases)

"unstable"



↳ geos generally diverge →
diverge in one direction or the other: s increasing or decreasing.

surface of revolution



stable geo ← maximum distance from axis (positive curvature)

unstable geo ← minimal distance from axis (negative curvature)

stable geo

torus