8.1–8.2

1

$\mathbb{R}^n$, $\{x^i\}$ Cartesian, $\{\bar{x}^i\}$ general

parametrized curve $\quad c(\lambda) = \langle c^1(\lambda), \ldots, c^n(\lambda)\rangle$, $\quad c^i(\lambda) = x^i \circ c(\lambda)$

calc 3 tangent vector $\quad c'(x) = \langle c^{1\prime}(\lambda), \ldots, c^{n\prime}(\lambda)\rangle$

$$c'(\lambda) = c^{i\prime}(\lambda) \left.\frac{\partial}{\partial x^i}\right|_{c(\lambda)}$$

"coordinate" independent

$$= \underbrace{c^{i\prime}(\lambda) \frac{\partial \bar{x}^j}{\partial x^i}(c(\lambda))}_{\bar{c}^{i\prime}(\lambda) = \frac{d}{d\lambda}(\bar{x}^i \circ c(\lambda)) \text{ chain rule.}} \left.\frac{\partial}{\partial \bar{x}^j}\right|_{c(\lambda)}$$

$$= \bar{c}^{j\prime}(\lambda) \left.\frac{\partial}{\partial \bar{x}^j}\right|_{c(\lambda)}$$

now work in general coordinates

ALONG CURVE:

covariant derivative function $f$

$$\nabla_{c'(\lambda)} f = c'(\lambda) f = c^{i\prime}(\lambda) \left.\frac{\partial f}{\partial x^i}\right|_{c(\lambda)}$$

sloppy: identify $c^i(\lambda) = x^i(\lambda)$

$$= x^{i\prime}(\lambda) \left.\frac{\partial f}{\partial x^i}\right|_{c(\lambda)}$$

$$= \frac{dx^i}{d\lambda}(\lambda) \left.\frac{\partial f}{\partial x^i}\right|_{c(\lambda)} = \frac{df}{d\lambda}(c(\lambda))$$

chain rule derivative

$\downarrow$

extend to vector field $Y$

$$[\nabla_{c'(\lambda)} Y]^i = Y^i{}_{;j}(c(\lambda))\, c^{j\prime}(\lambda)$$

$$= [Y^i{}_{,j}(c(\lambda)) + \Gamma^i{}_{jk}(c(\lambda))\, Y^k(c(\lambda))]\, c^{j\prime}(\lambda)$$

$$= \underbrace{Y^i{}_{,j}(c(\lambda))\, c^{j\prime}(\lambda)}_{\frac{\partial Y^i(c(\lambda))}{\partial x^j}\frac{dx^j(\lambda)}{d\lambda}} + \Gamma^i{}_{jk}(c(\lambda))\, c^{j\prime}(\lambda)\, Y^k(c(\lambda))$$

CHAIN RULE

$$= \frac{dY^i(c(\lambda))}{d\lambda} + \Gamma^i{}_{jk}(c(\lambda))\, c^{j\prime}(\lambda)\, Y^k(c(\lambda))$$

$$\equiv \frac{DY^i(c(\lambda))}{d\lambda}$$

only depends on value of $Y$ along $c(\lambda)$
does not require $Y$ to be a field.

intrinsic derivative along the curve

$Y^i(c(\lambda))$ field evaluated on curve

$\downarrow$

$Y^i(\lambda)$ field only defined on the curve

$$\boxed{\frac{DY^i}{d\lambda}(\lambda) = \frac{dY^i(\lambda)}{d\lambda} + \Gamma^i{}_{jk}(c(\lambda))\, c^{j\prime}(\lambda)\, Y^k(\lambda)}$$

$$\frac{DY^i}{d\lambda} = \frac{dY^i}{d\lambda} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}Y^k$$ suppress functional dependence

$$\downarrow$$

$$\frac{DT^{i\ldots}{}_{j\ldots}}{d\lambda} = \frac{dT^{i\ldots}{}_{j\ldots}}{d\lambda} + \Gamma^i{}_{\ell m}\frac{dx^\ell}{d\lambda}T^{m\ldots}{}_{j\ldots} + \cdots - \Gamma^m{}_{\ell j}T^{i\ldots}{}_{m\ldots} - \cdots$$

$$\left(= \nabla_{c'(\lambda)}T^{i\ldots}{}_{j\ldots} \text{ if } T \text{ is a field defined on } \mathbb{R}^n\right)$$

<u>parallel transport</u>

$$0 = \frac{DY^i}{d\lambda} = \frac{dY^i}{d\lambda} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}Y^k$$ system of 1st order ordinary differential eqns

$$Y^i(0) = Y^i_0$$ initial conditions

initial value problem has unique solution

allows us to keep a vector "constant" in any coord system in flatspace

or define what it means to keep it covariant constant in a curved space

orthogonal coords $\longrightarrow$ ON frame $\{e_{\hat i}\}$

$$\frac{DY^{\hat i}}{d\lambda} = \frac{dY^{\hat i}}{d\lambda} + \underbrace{\Gamma^{\hat i}{}_{\hat j\hat k}\,\omega^{\hat j}(c')}_{\hat\omega^i{}_j(c')}\,Y^{\hat k}$$

$\hat\omega^i{}_j(c')$ antisymmetric since $e_{\hat i}$ can only rotate

$$\hat\omega^i{}_j = \Gamma^{\hat i}{}_{\hat k\hat j}\,\omega^{\hat k} = \Gamma^{\hat i}{}_{k\hat j}\,dx^k$$

for parallel transport along coord lines

$$\underline{\hat\omega} = \hat\omega_k dx^k$$

only one 1-form component contributes $\quad x^k = x^k_0 + \lambda\delta^k_j \quad$ j fixed

$$\frac{dx^k}{d\lambda} = \delta^k{}_j$$ only one component

if $\hat\omega_j = 0$ for a given $j$,
frame parallel transported along that coord line.

$\boxed{8.1\text{–}8.2}$
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cylindrical coords: $(\rho, \phi, z)$

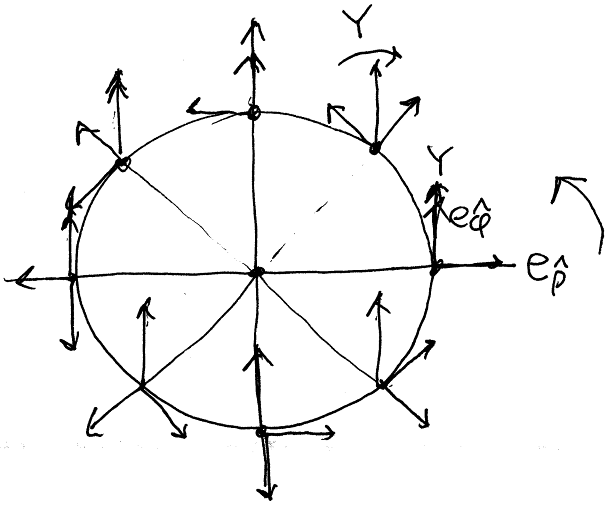
$$\hat{\underline{\omega}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d\phi$$

$$C: \quad \begin{aligned} \rho &= \rho_0 \\ \phi &= \phi_0 + \lambda \\ z &= z_0 \end{aligned} \qquad \langle c'^{i} \rangle = \langle 0, 1, 0 \rangle$$

$$Y: \quad \langle Y^{\hat{\rho}}, Y^{\hat{\phi}}, Y^{\hat{z}} \rangle$$

$$\frac{d\hat{\underline{Y}}}{d\lambda} = - \overbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\text{counterclockwise rotation of frame}} \hat{\underline{Y}}$$

clockwise rotation of parallel transported vector relative to frame to compensate




looking down z-axis

$$\hat{Y}(\lambda) = - \begin{bmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Y_0^{\hat{\rho}} \\ Y_0^{\hat{\phi}} \\ Y_0^{\hat{z}} \end{bmatrix}$$

$\boxed{8.1\text{–}8.2}$ spherical coords: $(r,\theta,\varphi)$

4

$$\hat{\underline{\omega}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d\theta + \underbrace{\begin{array}{c} \begin{array}{ccc} r & \theta & \varphi \end{array} \\ \begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & -\cos\theta \\ \sin\theta & \cos\theta & 0 \end{bmatrix} \end{array}}_{\underline{B}^{-1}\underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{rotation about z-axis}} \underline{B}} d\phi$$

$= \underline{B}^{-1} d\underline{B}$

$\underline{B} = \langle \vec{e}_{\hat r}, \vec{e}_{\hat\theta}, \vec{e}_{\hat\varphi} \rangle$

cartesian components

transforms to new frame

## 2-sphere geometry

for $r = r_0$, only $(\theta,\varphi)$ block of matrices needed.

$$\hat\omega = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cos\theta \, d\phi \qquad \hat{Y} = \begin{bmatrix} Y^{\hat\theta} \\ Y^{\hat\phi} \end{bmatrix}$$

frame parallel transported along $\theta$ lines

along $\phi$ circles

$\theta = \theta_0$

$\phi = \lambda$

$$\frac{d\hat{\underline{Y}}}{d\lambda} = -\frac{\hat\omega}{d\phi}\hat{Y} = -\cos\theta_0 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \underline{\hat{Y}}$$

$$\begin{bmatrix} Y^{\hat\theta}(\lambda) \\ Y^{\hat\phi}(\lambda) \end{bmatrix} = e^{\overbrace{-\lambda\cos\theta_0}^{\Delta\phi} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \begin{bmatrix} Y^{\hat\theta}_0 \\ Y^{\hat\varphi}_0 \end{bmatrix} = \begin{bmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{bmatrix} \begin{bmatrix} Y^{\hat\theta}_0 \\ Y^{\hat\varphi}_0 \end{bmatrix}$$

$\varphi = \lambda$

$\lambda = 0 \Leftrightarrow \phi = 0$
initial point
$\lambda = 2\pi \quad \phi = 2\pi$
final point after 1 revolution

defines vector field on sphere
once pick initial value at $\lambda = 0 = \varphi \rightarrow \quad \Delta\phi = \lambda\cos\theta_0$

example: $\hat{Y}(0) = e_{\hat\varphi}(\theta_0, 0)$ horizontal unit vector

$\langle Y^{\hat\theta}(0), Y^{\hat\varphi}(0) \rangle = \langle 0, 1 \rangle$

$\langle Y^{\hat\theta}(\varphi), Y^{\hat\varphi}(\varphi) \rangle = \langle \sin(\phi\cos\theta_0), \cos(\phi\cos\theta_0) \rangle \quad \theta_0 \rightarrow \theta$

$$Y = \sin(\phi\cos\theta_0)\, e_{\hat\theta} + \cos(\phi\cos\theta_0)\, e_{\hat\varphi}$$

can plot this on sphere. (Maple worksheet)

$\boxed{8.1\text{-}8.2}$

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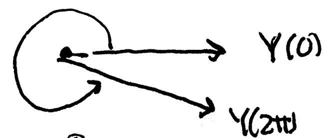
<u>upper hemisphere  $\cos\theta_0 > 0$</u>

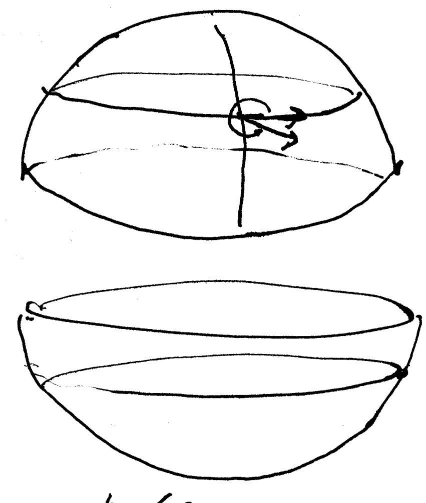
<u>near equator</u>

nearly return to original direction




$-2\pi\cos\theta_0$

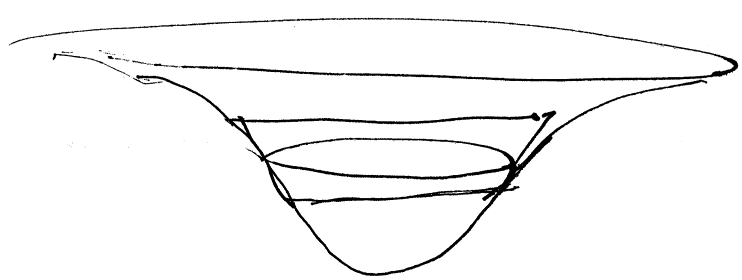
$$\frac{+2\pi}{2\pi(1-\cos\theta_0)} < 0$$



$\cos\theta_0 < 0$

opposite sign, direction ∥ prograde advances

compare tangent cone to circular orbit

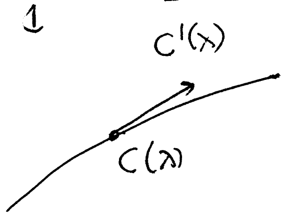


black hole : spin direction precesses in same sense as angular velocity

2/3 "geodetic effect" = spatial geometry of deformed equatorial plane of circular orbit

## $\boxed{8.3}$ Geodesics = autoparallel curves

1




$$\frac{DC'(\lambda)}{d\lambda} = 0$$ tangent parallel translated along curve

in coordinates: $x^i(\lambda) = x^i(C(\lambda))$

$$\frac{dx^i}{d\lambda} = u^i = \frac{Dx^i}{d\lambda} \quad \text{(scalars)}$$

$$\frac{Du^i}{d\lambda} = \frac{du^i}{d\lambda} + \Gamma^i{}_{kj} u^k u^j = 0$$

$$\boxed{\frac{D^2 x^i}{d\lambda} = \frac{d^2 x^i}{d\lambda^2} + \Gamma^i{}_{jk} \frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0}$$

$x^i(0) = x^i_0, \quad u^i(0) = u^i_0$

} initial value problem for 2nd order differential equations.

$\rightarrow$ unique soln.

length preserved:

$$u^i u_i = g_{ij} u^i u^j$$

$$\frac{d}{d\lambda}(u^i u_i) = \frac{D}{d\lambda}(u^i u_i) = \underbrace{\frac{Dg_{ij}}{d\lambda}}_{=0} u^i u^j + 2 g_{ij} u^i \underbrace{\frac{Du^j}{d\lambda}}_{=0} = 0$$

THIS IS AN AFFINE PARAMETRIZATION (special)

If we change parametrization:

$$\frac{dx^i}{d\bar\lambda} = \frac{dx^i}{d\lambda} \Big/ \frac{d\bar\lambda}{d\lambda} = \frac{u^i}{d\bar\lambda / d\lambda}$$ ← if not constant length of $\bar u^i = \frac{dx^i}{d\lambda}$ changes!

$$0 = \frac{D^2 x^i}{d\lambda^2} = \frac{d}{d\lambda}\left(\frac{dx^i}{d\lambda}\right) + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}$$

$$= \frac{d\bar\lambda}{d\lambda}\frac{d}{d\bar\lambda}\left(\frac{d\bar\lambda}{d\lambda}\frac{dx^i}{d\bar\lambda}\right) + \Gamma^i{}_{jk}\left(\frac{dx^j}{d\bar\lambda}\frac{d\bar\lambda}{d\lambda}\right)\left(\frac{dx^k}{d\bar\lambda}\frac{d\bar\lambda}{d\lambda}\right)$$

$$= \left(\frac{d\bar\lambda}{d\lambda}\right)^2 \frac{d^2 x^i}{d\bar\lambda^2} + \underbrace{\left(\frac{d\bar\lambda}{d\lambda}\frac{d}{d\bar\lambda}\frac{d\bar\lambda}{d\lambda}\right)}_{\frac{d^2\bar\lambda}{d\lambda^2}} \frac{dx^i}{d\bar\lambda} + \Gamma^i{}_{jk}\frac{dx^j}{d\bar\lambda}\frac{dx^k}{d\bar\lambda}\left(\frac{d\bar\lambda}{d\lambda}\right)^2$$

$$= \left(\frac{d\bar\lambda}{d\lambda}\right)^2 \left[\frac{D^2 x^i}{d\bar\lambda^2}\right] + \frac{d^2\bar\lambda}{d\lambda^2}\frac{dx^i}{d\bar\lambda}$$

$$\frac{D^2 x^i}{d\bar\lambda^2} = -\frac{d^2\bar\lambda / d\lambda^2}{(d\bar\lambda / d\lambda)^2}\frac{dx^i}{d\bar\lambda} \propto \frac{dx^i}{d\bar\lambda} = u^i$$

length changing but direction fixed

unless $\frac{d^2\bar\lambda}{d\lambda^2} = 0 \leftrightarrow \bar\lambda = a\lambda + b$ only linear change of parametrization allowed for simple geodesic equations "affine freedom" "arclength" parametrization (special affine parametrization)

$u^i u_i = \text{sgn}(u)\, \underbrace{|u|^2}_{\text{if } \neq 0} \rightarrow \hat{u}^i = \frac{u^i}{|u|} \qquad \hat{u}^i \hat{u}_i = \text{sgn}(u)$ unit tangent vector

$|u|$: constant

$$\| \quad \frac{dx^i}{d\lambda}\frac{dx_i}{d\lambda} = \text{sgn}(u)\frac{ds^2}{d\lambda^2}$$

$$ds^2 \equiv \text{sgn}(u)\, dx^i dx_i = \text{sgn}(u)\, g_{ij}\, dx^i dx^j$$

↳ to make nonnegative.

$$\hat{u}^i = \frac{dx^i/d\lambda}{ds/d\lambda} = \frac{dx^i}{ds}$$

"arclength parametrization" ⟶ unit tangent.

## examples

- Flat space — cartesian coords: $\frac{D^2 x^i}{d\lambda^2} = \frac{d^2 x^i}{d\lambda^2} = 0 \rightarrow x^i = a^i\lambda + b^i$

straight line solutions, linear functions of affine parameter

$$\frac{dx^i}{d\lambda} = a^i \qquad \left(\frac{dx^i}{d\lambda}\right)\left(\frac{dx_i}{d\lambda}\right) = a^i a_i = \text{sgn}(a)\,\underbrace{|a|^2}$$

if $= 1 \Leftrightarrow$ arclength parameter

then $b^i$ = freedom of referencept from which arclength is measured.

- $\mathbb{R}^3$ spherical coordinates: radial coord lines

$(x^1, x^2, x^3) = (r, \theta, \varphi)$ $\qquad r(\lambda) = r_0 + \lambda, \quad \theta(\lambda) = \theta_0, \quad \varphi(\lambda) = \varphi_0$

$$x^i(\lambda) = \delta^i{}_r \lambda + x^i_0$$

$$\frac{dx^i}{d\lambda} = \delta^i{}_r \qquad \frac{d^2x^i}{d\lambda^2} = 0$$

$c'(\lambda) = e_r$

unit vector parallel transported along r coord lines

$$\frac{D^2 x^i}{d\lambda^2} = \underbrace{\frac{d^2 x^i}{d\lambda^2}}_{=0} + \underbrace{\Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}}_{\Gamma^i{}_{rr} = 0} = 0$$

$\underline{\omega_r = 0}$

frame parallel transported along r coord lines.

$\boxed{8.3}$
3

"conserved quantity" $\Leftrightarrow$ constant along curve
"constant of the motion"

physics language — math translation

symmetries of metric lead to these

$$0 = (\mathcal{L}_\xi g)_{ij} = \cdots = \xi_{i;j} + \xi_{j;i}$$

Killing vector fields generate symmetries of metric

<u>example</u> $\mathbb{R}^3$, cartesian coords:

$P_i = \partial_i$ generate translations "linear momentum operator"

$L_i = \epsilon_{ijk} x^j \partial_k$ generate rotations "angular momentum operator"

momentum in familiar sense:

$$P(\xi) = \xi_i u^i = \xi_i \frac{dx^i}{d\lambda}$$

= component of tangent along Killing vector field.
"momentum" corresponding to $\xi$

$$\mathbb{R}^3: \quad P(P_j) = \partial_j \cdot \frac{dx}{d\lambda} = \delta_j^k \delta_{ki} \frac{dx^i}{d\lambda} = \delta_{ji} \frac{dx^i}{d\lambda} = \frac{dx_j}{d\lambda} = m\frac{dx_j}{dt} = mV_i = P_i$$

$$P(L_i) = \epsilon_{ijk} x^j \frac{dx^k}{d\lambda} = \left(\vec{X} \times \frac{d\vec{X}}{d\lambda}\right)_i = [m\vec{X} \times \vec{V}]_i = [\vec{X} \times \vec{P}]_i$$

If $\lambda = \frac{t}{M}$ ← time, ← mass

linear momentum & angular momentum
as we learned in high school.

$\boxed{\text{Symmetry leads to conserved quantities}}$ (along geodesics)

$$\frac{d}{d\lambda} P(\xi) = \frac{D}{d\lambda} P(\xi) = \frac{D}{d\lambda}\left(\xi_j \frac{dx^j}{d\lambda}\right) = \frac{D}{d\lambda}\left(\underbrace{g_{ji}}_{g_{ij}} \xi^i \frac{dx^j}{d\lambda}\right)$$

$$= \underbrace{\left(\frac{D}{d\lambda} g_{ij}\right)}_{0} \xi^i \frac{dx^j}{d\lambda} + g_{ij} \underbrace{\frac{D\xi^i}{d\lambda}}_{\left(\xi^i{}_{;k}\frac{dx^k}{d\lambda}\right)} \frac{dx^j}{d\lambda} + g_{ij} \xi^i \underbrace{\frac{D^2 x^j}{d\lambda^2}}_{0}$$

$$g_{ij} \xi^i{}_{;k} \frac{dx^k}{d\lambda} \frac{dx^j}{d\lambda} = \xi_{j;k} \frac{dx^k}{d\lambda}\frac{dx^j}{d\lambda}$$

$$= \underbrace{\xi_{(j;k)}}_{=0 \text{ Killing condition.}} \frac{dx^k}{d\lambda}\frac{dx^j}{d\lambda}$$

non-geodesic curves ( motion in physics sense)

$$\frac{D^2x^i}{d\lambda^2} = F^i \leftarrow \text{"force field"}$$

repeat derivation

$$\frac{d}{d\lambda} P(\xi) = \frac{D}{d\lambda}\left(\xi_i \frac{dx^i}{d\lambda}\right) = \cdots = 0 + 0 + \xi_i \frac{D^2x^i}{d\lambda^2} = \xi_i F^i \overset{?}{=} 0$$

momentum conserved if force field is orthogonal to killing vector field.

conservative force field : $F = -dU \rightarrow F_i = -\partial_i U$

$F^i = -g^{ij} U_{,j}$

$$\frac{d}{d\lambda} P(\xi) = \xi_i(-g^{ij} U_{,j}) = -\xi^j U_{,j} = -\xi U = -\mathcal{L}_\xi U \overset{?}{=} 0$$

if $U$ is invariant under transformations generated by $\xi$, corresponding momentum is conserved.

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Example $\quad m \frac{D^2x^i}{d\tau^2} = q F^i{}_j \frac{dx^j}{d\tau} \leftarrow$ proper time. electromagnetic field.

mass, acceleration, charge, velocity $u^j$ = unit 4-velocity

$$\frac{D^2x^i}{d\tau^2} = \frac{q}{m} F^i{}_j \frac{dx^j}{d\tau} \leftrightarrow \frac{Du^i}{d\tau} = \frac{q}{m} F^i{}_j u^j .$$

1) $\frac{D}{d\lambda}(u_i u^i) = 2u_i \frac{Du^i}{d\lambda} = 2u_i\left(\frac{q}{m} F^i{}_j u^j\right) = \frac{2q}{m} F_{ij} u^i u^j = 0$

electromagnetic field only rotates/pseudorotates 4-velocity remains unit vector

2) $\frac{d}{d\lambda} P(\xi) = \xi_i \underbrace{\left(\frac{q}{m} F^i{}_j u^j\right)}_{\text{Lorentz force}} = \frac{q}{m} F_{ij} \xi^i u^j = 0$ iff Lorentz force is orthogonal to killing vector field.

8.3

5

$$\frac{DU^i}{d\lambda} = \frac{dU^i}{d\lambda} + \Gamma^i{}_{jk} U^j U^k = 0$$

index position does not matter.

$$\frac{DU_i}{d\lambda} = \frac{D(g_{ij}U^j)}{d\lambda} = g_{ij} \underbrace{\frac{DU^j}{d\lambda}}_{=0} = 0$$

$\updownarrow$

$$\frac{DU_i}{d\lambda} = \frac{dU_i}{d\lambda} - U_k \Gamma^k{}_{ji} U^j = 0$$

$$= \frac{dU_i}{d\lambda} - \Gamma_{kji} U^k U^j = \frac{dU_i}{d\lambda} - \Gamma_{(kj)i} U^k U^j$$

$$\Gamma_{ijk} = \tfrac{1}{2}( g_{ij,k} - g_{jk,i} + \underbrace{g_{ki,j}}_{g_{ik,j}})$$

antisym in $(i,j)$

$$\Gamma_{(ij)k} = \tfrac{1}{2} g_{ij,k}$$

$$0 = \frac{dU_i}{d\lambda} - \tfrac{1}{2} g_{jk,i} U^j U^k$$

$$= \frac{d}{d\lambda}\underbrace{(g_{ij}U^j)}_{\frac{\partial}{\partial U^i}(\frac{1}{2} g_{jk}U^jU^k)} - \tfrac{1}{2} g_{jk,i} U^j U^k$$

$\leftarrow$ function of $(x^i, U^i)$
introduce: $\frac{\partial}{\partial x^i}, \frac{\partial}{\partial U^i}$

$$= \frac{d}{d\lambda}\frac{\partial}{\partial U^i}\left(\tfrac{1}{2} g_{jk}U^jU^k\right) - \frac{\partial}{\partial x^i}\left(\tfrac{1}{2} g_{jk}U^jU^k\right)$$

$$= \boxed{\frac{d}{d\lambda}\left(\frac{\partial T}{\partial U^i}\right) - \frac{\partial T}{\partial x^i} = 0}$$

$$T(x^i, U^i) = \tfrac{1}{2} g_{ij}U^iU^j = \tfrac{1}{2} U_i U^i$$

half self **inner** product

$U^i = \frac{dx^i}{dt}$ velocity

"Lagrange equations"

"kinetic energy"
= function on space $(x^i, \frac{dx^i}{dt})$ of positions & velocities

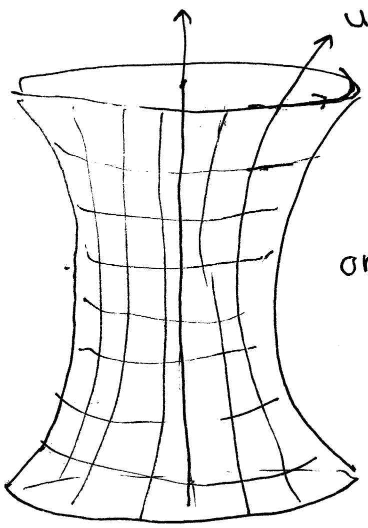
Lagrangian

need "calculus of variations" to understand meaning.

$\boxed{\text{geodesics extremize arclength}}$

Euclidean signature geometries
"shortest path between 2 points"

# surfaces of revolution



u "meridians" (vertical)

$\varphi$ "parallels" circles (horizontal)

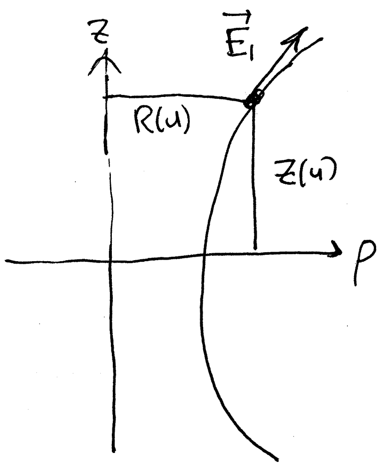
orthogonal grid

cylindrical coords

$\langle x, y, z\rangle = \langle \rho\cos\varphi, \rho\sin\varphi, z\rangle$

$\varphi \downarrow \theta$

for comparison with polar coords $(r, \theta)$ in plane




profile curve rotated around z-axis

surface: $\rho = R(u),\ z = Z(u)$

$$\vec{r} = \langle x, y, z\rangle = \langle R(u)\cos\theta, R(u)\sin\theta, Z(u)\rangle$$

$$\vec{E}_1 = \frac{\partial\vec{r}}{\partial u} = \langle R'(u)\cos\theta, R'(u)\sin\theta, Z'(u)\rangle$$

$$\vec{E}_2 = \frac{\partial\vec{r}}{\partial\theta} = \langle -R(u)\sin\theta, R(u)\cos\theta, 0\rangle$$

$$g_{uu} = g_{11} = \vec{E}_1\cdot\vec{E}_1 = R'(u)^2 + Z'(u)^2$$

$$g_{uv} = g_{12} = \vec{E}_1\cdot\vec{E}_2 = 0$$

$$g_{\theta\theta} = g_{22} = \vec{E}_2\cdot\vec{E}_2 = R(u)^2$$

$$ds^2 = d\vec{r}\cdot d\vec{r} = \left(\frac{\partial\vec{r}}{\partial u}du + \frac{\partial\vec{r}}{\partial\theta}d\theta\right)\cdot\left(\frac{\partial\vec{r}}{\partial u}du + \frac{\partial\vec{r}}{\partial\theta}d\theta\right)$$

$$= \underbrace{[R'(u)^2 + Z'(u)^2]\,du^2}_{ds_u^2 = dr^2\ ??} + \underbrace{R(u)^2 d\theta^2}_{ds_\theta^2}$$

$$r = \int dr = \int\sqrt{R'(u)^2 + Z'(u)^2}\,du \qquad \int_0^{2\pi} ds_\theta = 2\pi R(u) = C(u)$$

circumferential radius

if integrable & invertible:

$$\boxed{ds^2 = dr^2 + R(r)^2 d\theta^2}$$

– easier to study.
– easier to present

$\left(\text{but we can still do this: } ds^2 = g_{uu}du^2 + R(u)^2 d\theta^2\right)$

coordinates on surface: $(x^1, x^2) = (u, \theta) \to (r, \theta)$

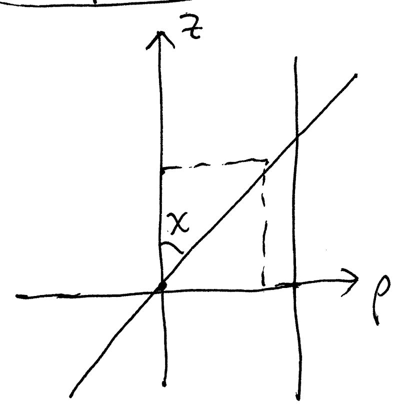
$(x^i)$, $i, j, k = 1, 2$; $u, \theta$; $r, \theta$ } context

orthogonal coordinate frame $\underbrace{\vec{E}_1, \vec{E}_2}_{\text{tangents to coordinate lines}} \leftrightarrow E_1, E_2 = \partial_1, \partial_2 = \partial_r, \partial_\theta$

$\boxed{8.4}$
2

# examples

## line profiles




plane:
$\langle R(u), Z(u)\rangle = \langle r, 0\rangle$ : : $R(r) = r$

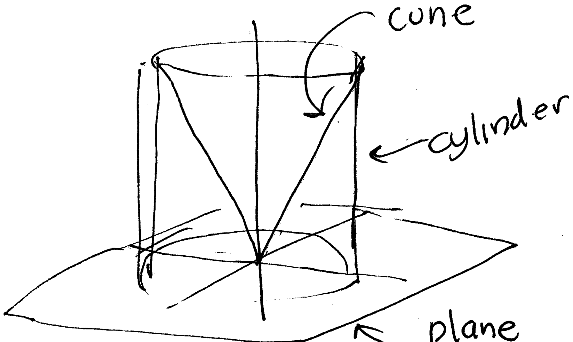
cylinder: $= \langle R_0, z\rangle$ : $R = R_0$, $Z = z$

~~r = R0~~

cone : $= \langle r\sin\chi, r\cos\chi\rangle \leftarrow \chi$ constant

$$ds^2 = dr^2 + \underbrace{r^2\sin^2\chi\, d\theta^2}$$

→ $r^2$ plane
→ $R_0^2$ cylinder

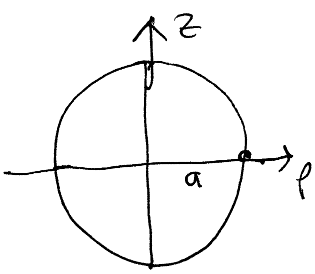


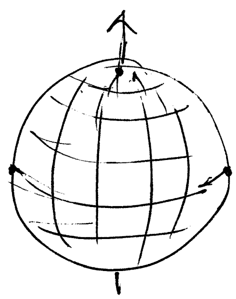

$$ds^2 = \underbrace{r_0^2\, d\theta^2}_{dr^2} + \underbrace{r_0^2\sin^2\theta\, d\varphi^2}_{r_0^2\sin^2\left(\frac{r}{r_0}\right)}$$

$r = r_0\theta$

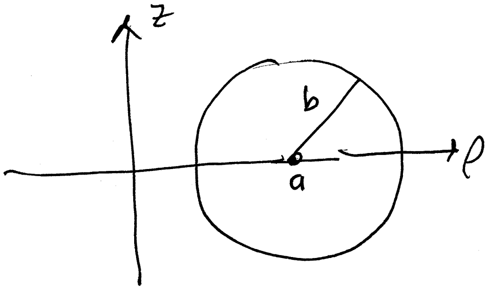
$$\boxed{R(r) = r_0 \sin\left(\frac{r}{r_0}\right)}$$

## circle profiles

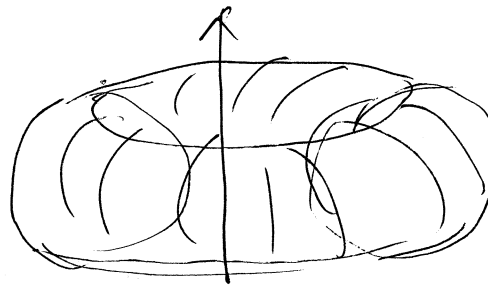


sphere




details later



donut = torus

$\boxed{8.4}$
3

connection

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{ii}(g_{ij,k} - g_{jk,i} + g_{ki,j}) \quad \cancel{\sum_i}$$

$g_{rr} = 1$
$g_{\theta\theta} = R(r)^2$

only $g_{\theta\theta,r} = 2R(r)R'(r) \neq 0$.

$$\Gamma^r{}_{\theta\theta} = -\frac{1}{2} g_{\theta\theta,r} = -R'(r)R(r)$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = R'(r)/R(r)$$

$$0 = \frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = \frac{du^i}{d\lambda} + \Gamma^i{}_{jk}u^j u^k \longrightarrow \begin{cases} \frac{d^2r}{d\lambda^2} = \dots \\ \frac{d^2\theta}{d\lambda^2} = \dots \end{cases}$$

tangent: $u = \frac{dr}{d\lambda}\frac{\partial}{\partial r} + \frac{d\theta}{d\lambda}\frac{\partial}{\partial\theta}$

Killing vector field $\xi = \partial_\theta$

conserved momentum $P(\partial_\theta) = \partial_\theta \cdot u = g_{\theta i}u^i = g_{\theta\theta}\frac{d\theta}{d\lambda}$

$= \boxed{R(r)^2 \frac{d\theta}{d\lambda} = \ell}$ angular momentum about z-axis

constant length tangent:

$$\frac{1}{2} g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}R(u)^2\underbrace{\left(\frac{d\theta}{d\lambda}\right)^2}_{\left(\frac{\ell}{R(r)}\right)^2} = \mathcal{E} \quad \text{"energy"}$$

eliminate $\frac{d\theta}{d\lambda}$

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{2R(r)^2} = \mathcal{E}}$$

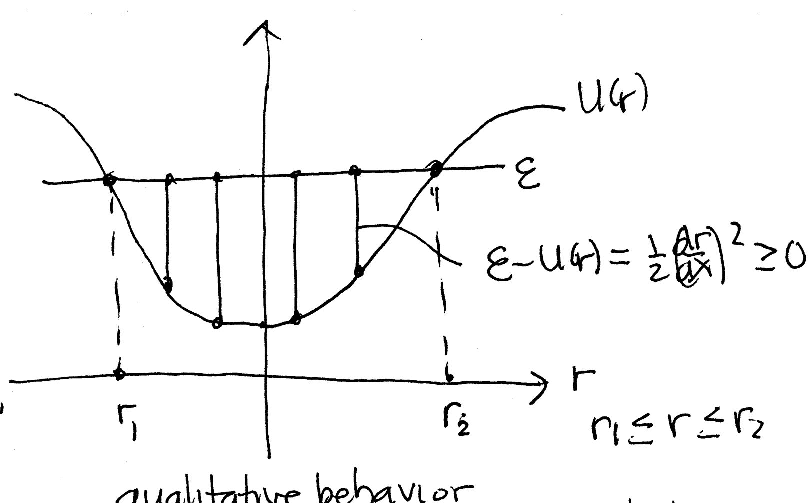
$$\hookrightarrow \frac{dr}{d\lambda} = \pm\sqrt{2\mathcal{E} - \ell^2/R(r)^2}$$

$$\lambda = \pm\int \frac{dr}{\sqrt{2\mathcal{E} - \ell^2/R(r)^2}} \quad ?$$

potential $U(r) = \frac{\ell^2}{2R(r)^2}$

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + U(r) = \mathcal{E}$$

$$\frac{d\theta}{d\lambda} = \frac{\ell}{R(r)^2} \longleftarrow$$




tracing out geodesic with $\lambda$ = "geodesic motion"

radial motion "in a potential"

qualitative behavior clear from shape of potential (see 8.5)

$\boxed{8.4}$

4

orthonormal frame:

$$e_{\hat r} = \partial_r \,,\quad e_{\hat\theta} = \frac{1}{R(r)}\partial_\theta \qquad \omega^{\hat r} = dr,\ \omega^{\hat\theta} = R(r)\,d\theta$$

$$[e_{\hat r}, e_{\hat\theta}] = e_{\hat r}(R(r)^{-1})\,\partial_\theta = -R(r)^{-2}R'(r)\,\partial_\theta = \underbrace{-(\ln R(r))'}_{C^{\hat\theta}{}_{\hat r\hat\theta}}\, e_{\hat\theta}$$

only nonzero Lie bracket function

$$\Gamma^{\hat i}{}_{\hat j\hat k} = \Gamma_{\hat i\hat j\hat k} = \tfrac12\left(C_{\hat i\hat j\hat k} - C_{\hat j\hat k\hat i} + C_{\hat k\hat i\hat j}\right)$$

$$\Gamma_{\hat r\hat\theta\hat\theta} = \tfrac12\left(\cancel{C_{\hat r\hat\theta\hat\theta}} - C_{\hat\theta\hat\theta\hat r} + C_{\hat\theta\hat r\hat\theta}\right) = C_{\hat\theta\hat r\hat\theta} \qquad \Gamma^{\hat r}{}_{\hat\theta\hat\theta} = C^{\hat\theta}{}_{\hat r\hat\theta}$$

$$\Gamma_{\hat\theta\hat\theta\hat r} = \tfrac12\left(C_{\hat\theta\hat\theta\hat r} - C_{\hat\theta\hat r\hat\theta} + \cancel{C_{\hat r\hat\theta\hat\theta}}\right) = -C_{\hat\theta\hat r\hat\theta} \qquad = -\Gamma^{\hat\theta}{}_{\hat\theta\hat r}$$

anti symmetric of course

$$\hat\omega = (\omega^{\hat i}{}_{\hat j}) = (\Gamma^{\hat i}{}_{\hat k\hat j}\,\omega^{\hat k}) = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\underbrace{(-(\ln R(r))')\underbrace{\omega^{\hat\theta}}_{R(r)d\theta}}_{=(-R'(r)\,d\theta)}$$

$$\nabla_{e_{\hat r}} e_{\hat r} = 0 = \nabla_{e_{\hat r}} e_{\hat\theta}$$

$$\nabla_{e_{\hat\theta}} e_{\hat r} = (\ln R(r))'\, e_{\hat\theta}$$

$$\nabla_{e_{\hat\theta}} e_{\hat\theta} = -(\ln R(r))'\, e_{\hat r}$$

$$\hat T = e_{\hat\theta} = \frac{dx^i}{ds}\partial_i$$ unit tangent to $\theta$ circles, $ds = R(r)\,d\theta$  $\qquad \nabla_{e_{\hat\theta}} = \frac{D}{ds}$ along $\theta$ circles

$$\frac{D\hat T}{ds} = K(r)\,\hat N \qquad\qquad \text{identify}\begin{cases} K(r) = |\ln R(r)|' \geq 0 \\ \hat N = -(\underbrace{\operatorname{sgn} R'(r)}_{\epsilon = \pm 1})\, e_{\hat r}\end{cases}$$

$$\frac{d\hat N}{ds} = -K(r)\,\hat T$$

intrinsic curvature of $\theta$ circles →

$R(r) = 1/K(r)$

intrinsic radius of curvature

$$\frac{De_{\hat\theta}}{ds} = -\underbrace{\epsilon K}\, e_{\hat r}\,,\qquad \frac{De_{\hat r}}{ds} = \underbrace{\epsilon K}\, e_{\hat\theta}$$

$$\frac{d\Phi}{ds} = \frac{\epsilon}{R} \qquad \frac{d\Phi}{ds} \longrightarrow \Phi = \epsilon K s = \frac{\epsilon}{R}s \qquad \frac{\text{arc}}{\text{radius}} = \text{angle}$$

but ↗

$\boxed{8.4}$
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along $\theta$: $ds = R(u)\, d\theta$

$$\frac{De_{\hat\theta}}{d\theta} = -\underbrace{\epsilon K R}\, e_{\hat r}\,, \quad \frac{De_{\hat r}}{d\theta} = \underbrace{\epsilon K R}\, e_{\hat\theta}$$

$$\frac{d\Phi}{d\theta} = \epsilon K R = R'(r) \quad \text{constant.}$$

$$\Phi = \underbrace{R'(r)}_{\equiv \Omega(r)}(\theta - \theta_0) \longrightarrow \theta_0 = 0\,;\ \Phi = \Omega\theta$$

$\theta$ angular velocity

$$\begin{pmatrix} \frac{De_{\hat r}}{d\theta} & \frac{De_{\hat\theta}}{d\theta} \end{pmatrix} = \begin{pmatrix} e_{\hat r} & e_{\hat\theta} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix} \Omega$$

---

parallel transport

$$0 = \frac{D}{d\theta}(\underbrace{X^{\hat r} e_{\hat r} + X^{\hat\theta} e_{\hat\theta}}_{(e_{\hat r}\ e_{\hat\theta})\begin{pmatrix} X^{\hat r} \\ X^{\hat\theta}\end{pmatrix}}) = \left(\frac{dX^{\hat r}}{d\theta} e_{\hat r} + \frac{dX^{\hat\theta}}{d\theta} e_{\hat\theta}\right) +$$

$$= (e_{\hat r}\ e_{\hat\theta}) \begin{pmatrix} dX^{\hat r}/d\theta \\ dX^{\hat\theta}/d\theta \end{pmatrix} + (e_{\hat r}\ e_{\hat\theta}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Omega$$

$$\frac{d}{d\theta}\begin{pmatrix} X^{\hat r} \\ X^{\hat\theta} \end{pmatrix} = \underbrace{-\Omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\Omega\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} X^{\hat r} \\ X^{\hat\theta} \end{pmatrix}$$

$$\begin{pmatrix} X^{\hat r} \\ X^{\hat\theta} \end{pmatrix} = e^{\Omega\theta\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} X_0^{\hat r} \\ X_0^{\hat\theta} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\Omega\theta & \sin\Omega\theta \\ -\sin\Omega\theta & \cos\Omega\theta \end{pmatrix}}_{\substack{\text{clockwise rotation} \\ \text{by } \Omega\theta \text{ if } \Omega > 0 \\ \text{as } \theta \text{ increases}}} \begin{pmatrix} X_0^{\hat r} \\ X_0^{\hat\theta} \end{pmatrix}$$

since frame rotating
by angle $\Omega\theta$ counterclockwise
(relative to parallel frame)

$\boxed{8.4}$
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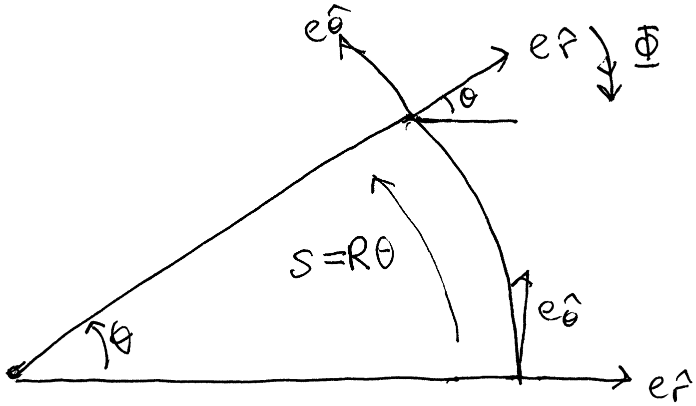



$\epsilon > 0$ $\quad e_{\hat r}$ rotates towards $e_{\hat\theta}$

$e_{\hat r}$ points outward

frame rotates forward by $\theta$
parallel transported vectors
rotate backwards to
compensate but
by different angle

$\Phi$

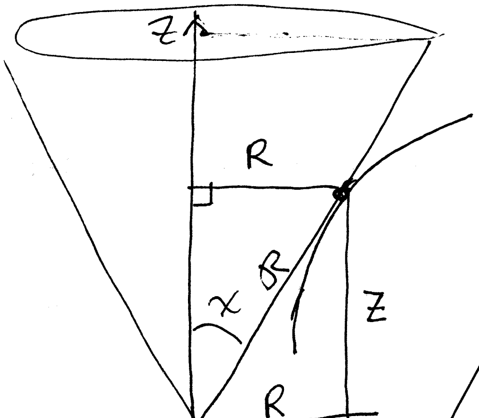
$\theta = \dfrac{s}{R}$ ← circumferential radius

eliminate $s$

$$\frac{d\Phi}{ds} = K = \frac{1}{\mathcal{R}} \rightarrow \Phi = \frac{s}{\mathcal{R}} = \frac{R}{\mathcal{R}}\,\theta$$

$\le 1$

intrinsic radius
falls short of
compensating




tangent cone.

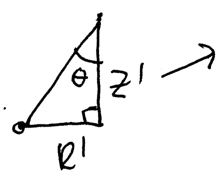
parallel transported
vectors
experience net
forward rotation
by angle

$$\theta - \phi = \theta\left(1 - \frac{R}{\mathcal{R}}\right)$$

↓ 1 revolution

$$\Delta\varphi = 2\pi\left(1 - \frac{R}{\mathcal{R}}\right) > 0$$

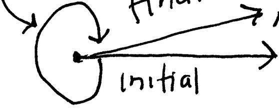
$$\sin\chi = \frac{R}{\mathcal{R}} = \underbrace{\frac{|R'|}{(R'^2 + z'^2)^{1/2}}}_{} \le 1$$




→ tangent condition

---

$\Delta\Phi < 2\pi$




$\Delta\varphi = 2\pi - \Delta\Phi > 0$

$\boxed{8.5}$
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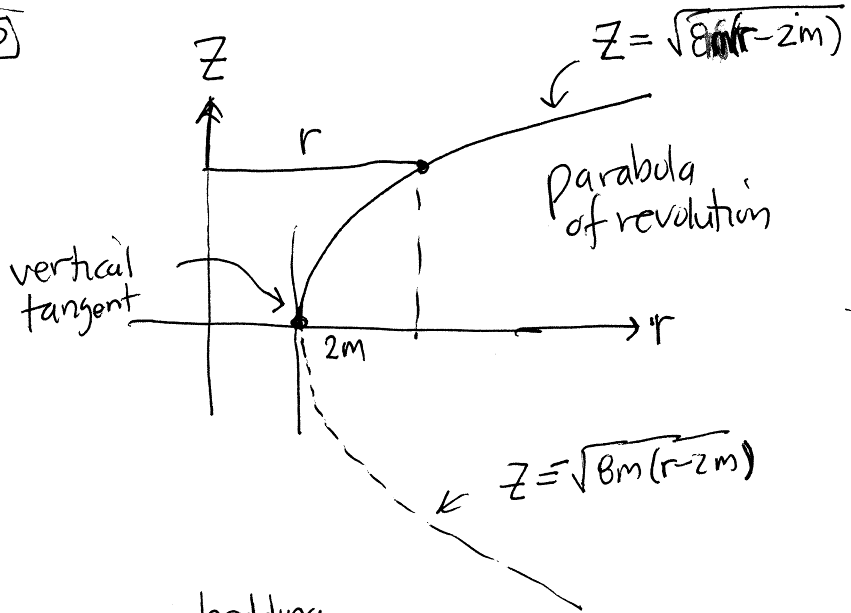
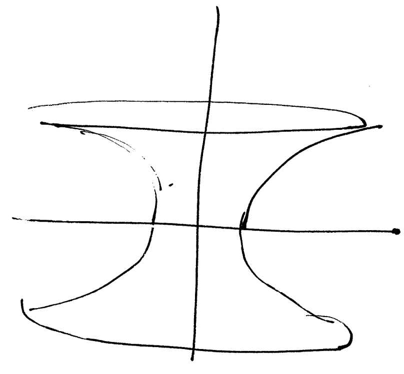


wormhole

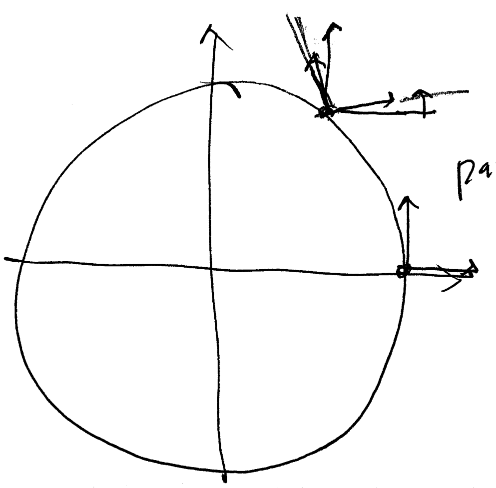
embedding diagram of intrinsic geometry of equatorial plane in black hole spacetime

circumferential radius

$$ds^2 = \frac{dr^2}{1-\frac{2m}{r}} + r^2 d\theta^2 \qquad r > 2m \qquad \rightarrow \text{calculate } \Omega, \Delta\varphi$$

$\uparrow$

$g_{rr} \neq 1$ need general approach



$r$–$\theta$ coordinate plane picture

parallel transport

prograde rotation of axes.

$\frac{2}{3}$ of "geodetic precession" of gyro in free fall circular orbit

is this effect.

$\boxed{8.5}$ geodesics as classical mechanics?

1

$\mathbb{R}^3$, cartesian coords $\{x^i\}$, metric: "$\delta_{ij}$"

mass x acceleration = force

$$m \frac{d^2x^i}{dt^2} = F^i = -\nabla^i U \qquad \text{conservative force field } U(\vec{x})$$

$$m \frac{d^2x^i}{dt^2} + \nabla^i U = 0 \qquad \leftarrow \text{math people like zeros!}$$

$$\downarrow$$

$$E \equiv \underbrace{\tfrac{1}{2} m \delta_{ij} \frac{dx^i}{dt}\frac{dx^j}{dt}}_{K} + U \qquad \text{kinetic + potential energy } \text{"}\tfrac{1}{2}mv^2\text{"}$$

$$\frac{dE}{dt} = \tfrac{1}{2} m \delta_{ij} \left(\frac{d^2x^i}{dt^2}\frac{dx^j}{dt} + \frac{dx^i}{dt}\frac{d^2x^j}{dt^2}\right) + \frac{d}{dt} U \qquad \text{chain rule: } U(\vec{x}) \leftarrow \vec{x}(t)$$

$$= m \delta_{ij} \frac{d^2x^i}{dt^2}\frac{dx^j}{dt} + \partial_j U \frac{dx^j}{dt}$$

$$= \Big(\underbrace{m \delta_{ij} \frac{d^2x^i}{dt^2} + \partial_j U}_{=0}\Big) \frac{dx^j}{dt} = 0$$

energy is conserved in time-independent potential.

"Freemotion" $\leftrightarrow$ $F^i = 0$ $\leftrightarrow$ speed constant. $\leftrightarrow$ $\frac{d^2x^i}{dt^2} = 0$ acceleration zero

true in any coordinates ⟶ $\frac{D^2x^i}{d\lambda^2} = 0$ zero covariant second derivative

$t \to \lambda$ affine parameter

set $m = 1$ (no need if no force)

$$K = \tfrac{1}{2} g_{ij} \frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = \mathcal{E} \quad \text{constant}$$

killing vector field $\mathcal{L}_\xi g_{ij} = \xi_{i;j} + \xi_{j;i} = 0$

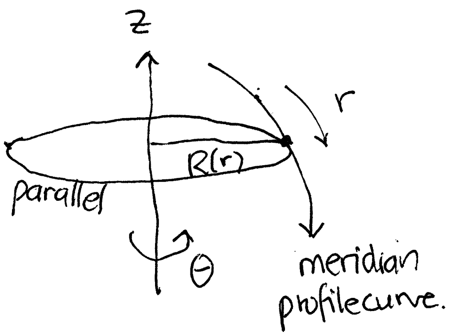
"constant of the motion" $\quad \xi_i \frac{dx^i}{d\lambda} = P(\xi)$ momentum component along $\xi$

GEODESIC MOTION

$\boxed{8.5}$ 2

## surface of revolution in $\mathbb{R}^3$

$$g = dr \otimes dr + \underbrace{R(r)^2}\, d\theta \otimes d\theta \quad \leftarrow R\,d\theta = dS_\theta\,,\ dr \text{ arclength}$$




$= r^2$ flat plane

$= r_0^2 \sin^2 \underbrace{\left(\frac{r}{r_0}\right)}_{\theta}$ sphere of radius $r_0$ $\qquad r = r_0\theta$ polar arclength coord.

$= r^2 \sin^2 \chi$ cone of opening angle $\chi$

etc.

$(x^i) = (r, \theta)$

$$\frac{D^2 x^i}{d\lambda^2} = \frac{d^2x^i}{d\lambda^2} + \Gamma^i{}_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0$$

$$\frac{d^2r}{d\lambda^2} + \Gamma^r{}_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2 = \frac{d^2r}{d\lambda^2} - R'R\left(\frac{d\theta}{d\lambda}\right)^2 = 0 \rightarrow \frac{d^2r}{d\lambda^2} = R'R\left(\frac{d\theta}{d\lambda}\right)^2 = \text{"}F^r\text{"}$$

effective radial force "centrifugal force"

$$\frac{d^2\theta}{d\lambda^2} + 2\Gamma^\theta{}_{r\theta}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} = \frac{d^2\theta}{d\lambda^2} + 2\,\frac{R'}{R}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} = 0$$

$$= R^{-2}\frac{d}{d\lambda}\underbrace{\left(R^2\frac{d\theta}{d\lambda}\right)}$$

$$\ell = \frac{\partial}{\partial\theta}\cdot \text{"}\left(\frac{d\vec{x}}{d\lambda}\right)\text{"}$$

conserved angular momentum around symmetry axis ①

$\boxed{\begin{array}{l}\text{general soln}\\ \text{4 arbitrary constants}\\ \text{initial data: } r_0, \theta_0, \left.\frac{dr}{d\lambda}\right|_0, \left.\frac{d\theta}{d\lambda}\right|_0\end{array}}$

$$\mathcal{E} = \frac{1}{2} g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}$$

$$= \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}R^2\left(\frac{d\theta}{d\lambda}\right)^2$$

② constant length of tangent vector

$\ell = R^2 \frac{d\theta}{d\lambda}$

$\frac{d\theta}{d\lambda} = \frac{\ell}{R^2} \leftarrow$ angular motion only depends on $R(r)$

eliminate angular velocity

2 constants of motion allow integration of 2nd order DEs to become 2 first order DEs.

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}R^2\left(\frac{\ell}{R^2}\right)^2$$

$$= \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{\ell^2}{2R(r)^2}}_{U_{eff}(r)}$$

"radial" motion along meridians decouples from angular motion.

$U_{eff}(r) \leftarrow$ "effective potential"

$$F_{eff} = -\frac{d}{dr}U_{eff}(r) = -(-2)\frac{\ell^2}{2}R^{-3}R' = \frac{\ell^2 R'}{R^3} = R'R\left(\frac{\ell}{R^2}\right)^2 = R'R\left(\frac{d\theta}{d\lambda}\right)^2 = \text{"}F^r\text{"}$$

really $\boxed{\text{centrifugal potential}}$

generates "fictitious" force

$\boxed{8.5}$
3

FLAT PLANE $R(r) = r$

$\frac{d\theta}{d\lambda}$ = angular velocity

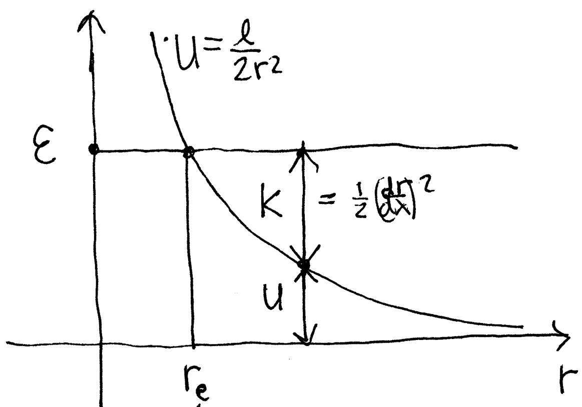
$r\frac{d\theta}{d\lambda}$ = velocity in angular direction

$\ell = r\left(r\frac{d\theta}{d\lambda}\right)$ = angular momentum about z-axis ($m=1$!)

$U = \frac{\ell^2}{2r^2}$ centrifugal potential

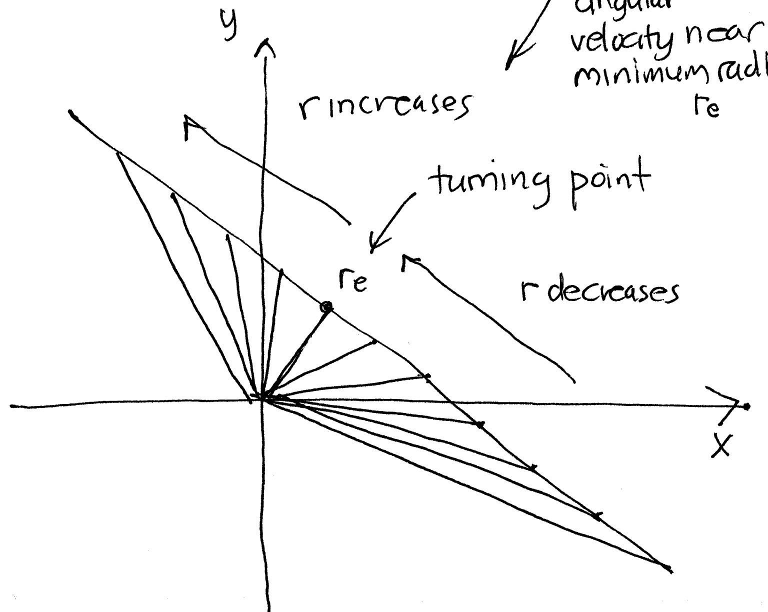
$\frac{d\theta}{d\lambda} = \frac{\ell}{r^2}$ small r fast $d\theta/d\lambda$

fastest angular velocity near minimum radius $r_e$




turning point (minimal radius)

$\frac{dr}{d\lambda} = 0$




geo = straight line traced out at constant speed

axis "protected" by a centrifugal potential "barrier"

cannot get to $r=0$ unless $\ell = 0$ (purely radial motion in 2-d) (aimed at origin)

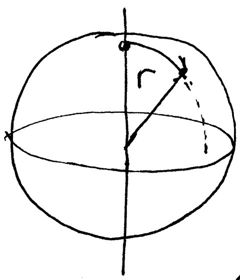
$\boxed{8.5}$

4

sphere of radius $r_0$

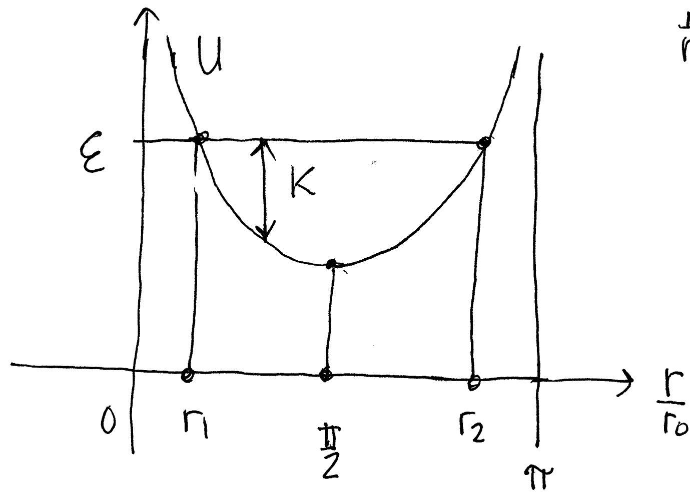
$R(r) = r_0 \sin\left(\frac{r}{r_0}\right)$

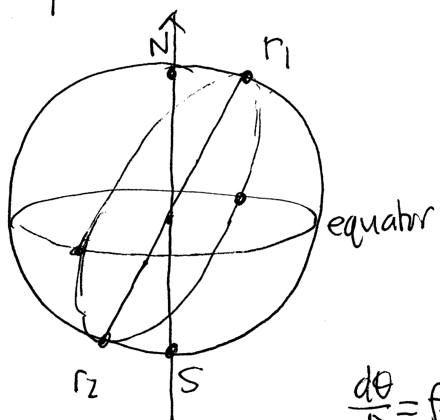
$U = \dfrac{\ell^2}{2r_0^2 \sin^2\left(\frac{r}{r_0}\right)} \to \infty$ at $\frac{r}{r_0} = 0, \pi$ poles




$\frac{r}{r_0}$ = polar angle

equator at $\frac{r}{r_0} = \frac{\pi}{2}$




tilted great circle
= geo

$\frac{d\theta}{d\lambda}$ = fastest at extremal radii , $\frac{dr}{d\lambda}$ fastest near equator

poles are protected by centrifugal potential "barriers"

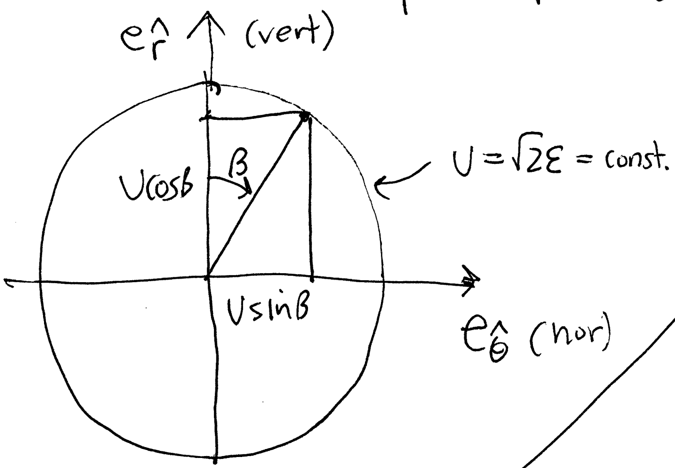
only $\ell = 0$ allows reaching $\frac{r}{r_0} = 0, \pi$
great circles
passing through poles.

$\left[ \ell = r_0^2 \underbrace{\sin^2\left(\frac{r}{r_0}\right)}_{=0 \text{ at poles}} \frac{d\theta}{d\lambda} \right]$

8.5

5

Orthonormal components of tangent = velocity




pular coords in tangent plane

$$ds^2 = dr^2 + R^2 d\theta^2$$ arclength

$$U^2 = \frac{ds^2}{d\lambda^2} = \left(\frac{dr}{d\lambda}\right)^2 + R^2\left(\frac{d\theta}{d\lambda}\right)^2 \leftarrow$$ speed$^2$ = length$^2$ (velocity)

$$\mathcal{E} = \frac{1}{2}U^2 = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{R^2}{2}\left(\frac{d\theta}{d\lambda}\right)^2$$

$$\underbrace{2\mathcal{E}}_{U^2} = \underbrace{\left(\frac{dr}{d\lambda}\right)}_{V^{\hat{r}}}{}^2 + \underbrace{\left(R\frac{d\theta}{d\lambda}\right)}_{V^{\hat{\theta}}}{}^2$$

$\downarrow$

"speed" $U = \sqrt{2\mathcal{E}}$

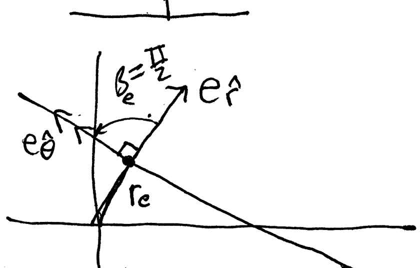
$$\langle V^{\hat{r}}, V^{\hat{\theta}}\rangle = \sqrt{2\mathcal{E}}\,\langle \cos\beta, \sin\beta\rangle$$

$$R\left[R\frac{d\theta}{d\lambda} = \sqrt{2\mathcal{E}}\sin\beta\right]$$

$$\ell = R^2\frac{d\theta}{d\lambda} = \sqrt{2\mathcal{E}}\,R(r)\sin\beta$$

$$\frac{\ell}{\sqrt{2\mathcal{E}}} = R(r)\sin\beta = \text{constant.}$$

flatplane




at turning point $\beta = \frac{\pi}{2}$

$r\sin\beta$ = constant

$\parallel$

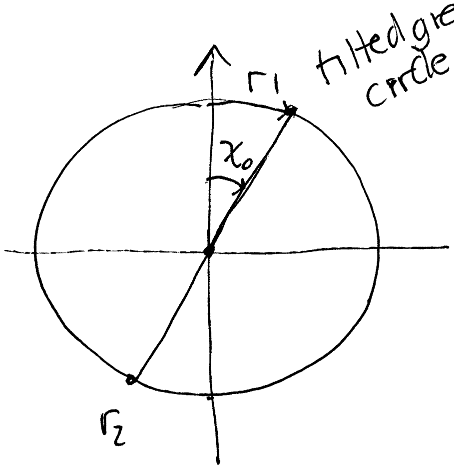
$r_e\sin\beta_e = r_e\sin\frac{\pi}{2} = r_e$

constant is minimal radius!

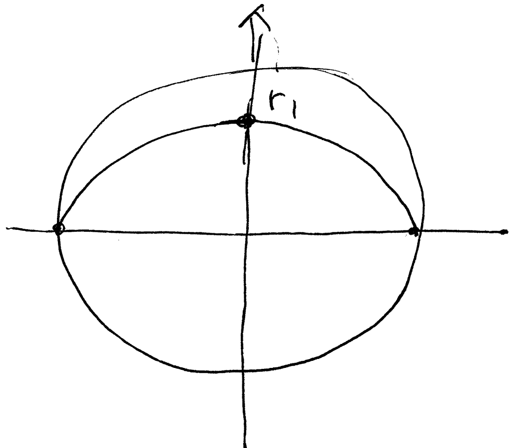
at turning points in general:

$\frac{dr}{d\lambda} = 0 \rightarrow \beta = \pm\frac{\pi}{2}$ purely $\theta$ motion

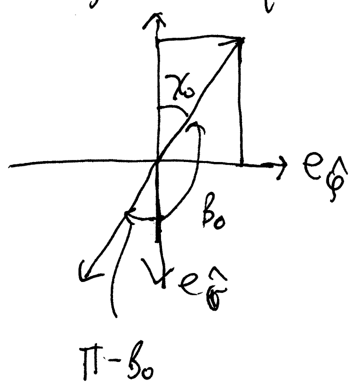
$\boxed{8.5}$
6




edge on at equator




perpendicular view to great circle diameter




initial data at equator $\frac{r}{r_0} = \frac{\pi}{2}$

$\beta_0, \pi-\beta_0 = \chi_0, \pi-\chi_0$

initial angle at equator $=$ tilt angle / complement.

$$r_0 \sin\left(\frac{r}{r_0}\right) \sin\beta = \text{const}$$

$$\parallel$$

$$r_0 \sin\frac{\pi}{2} \sin\beta_{equator} = r_0 \sin\beta_{equator}$$

$$\parallel \qquad\qquad\qquad\qquad \parallel$$

$$r_0 \sin\left(\frac{r_e}{r_0}\right) \underbrace{\sin\frac{\pi}{2}}_{=1} = \underbrace{r_0 \sin\left(\frac{r_e}{r_0}\right)}_{}$$

$\beta = \pm\frac{\pi}{2}$ at turning pt.

$R(r_e)$ = distance from axis. again.

---

$$\frac{\ell}{\sqrt{2E}} = R(r) \underbrace{\sin\beta}_{=1 \text{ at radial turning points}} \longrightarrow R(r_e)$$

at extremal radii

this combination of 2 constants of motion

has simple geometrical interpretation

8.5
7

arclength parametrized geodesics
for plane / sphere are relatively complicated

solutions more easily found using geometry

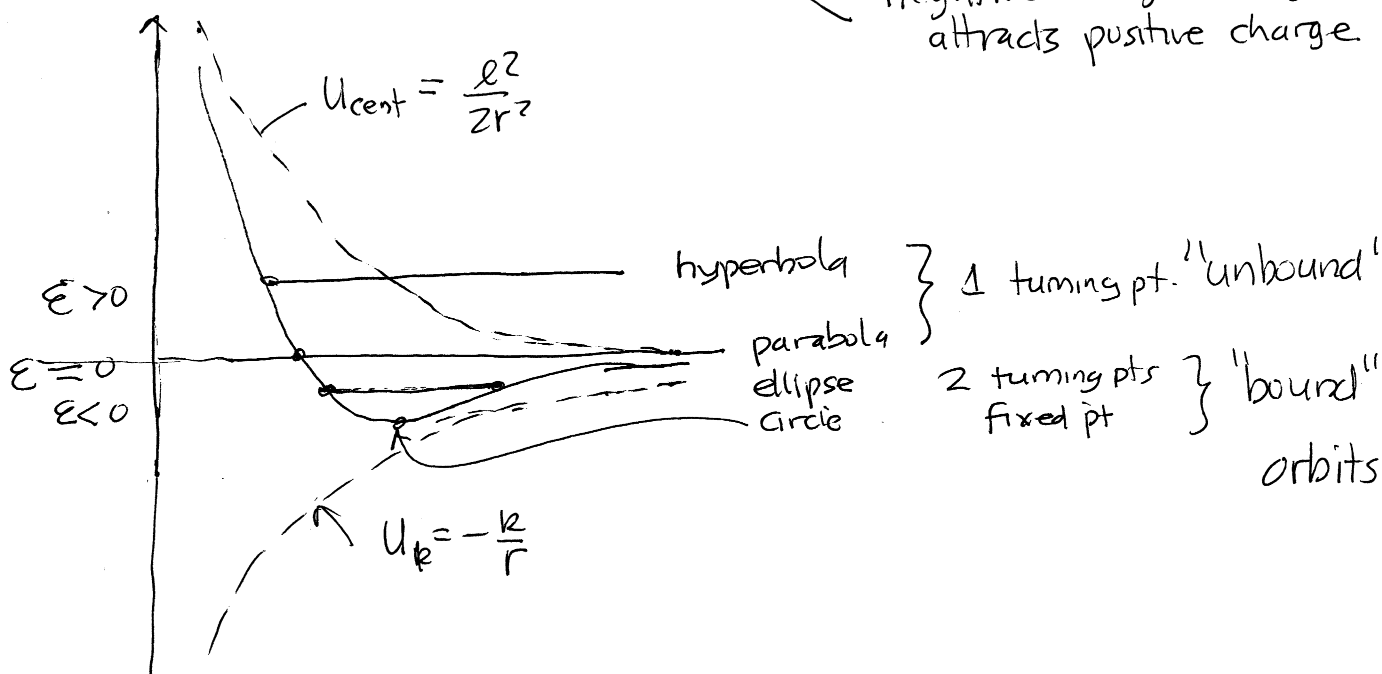
straight line, great circle → re-express in polar/spherical coordinates.

BUT
flat plane with rotationally symmetric potential = radial force field
easier than straight line motion. for inverse square force field.

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{r^2}{2}\left(\frac{d\theta}{d\lambda}\right)^2}_{\downarrow \; \frac{\ell^2}{2r^2}} - \frac{k}{r} \quad \longleftarrow \quad -\frac{d}{dr}\underbrace{\left(\frac{-k}{r}\right)}_{} = -\frac{k}{r^2} = F^r$$

attractive force if $k > 0$

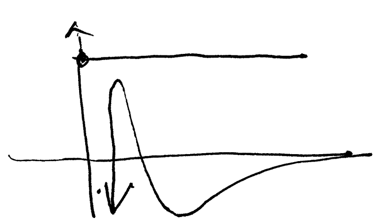
gravity,
negative charge at origin attracts positive charge.




add $-\frac{k_2}{r^3}$ term ⟷ exact G.R. orbits

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{2r^2} - \frac{k}{r} - \frac{k_2}{r^3}$$

cubed power wins near $r=0$
turns off centrifugal potential → capture orbits



# integrating first order DEs ?

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + U(r) = \mathcal{E} \qquad \frac{d\theta}{d\lambda} = \frac{\ell}{R(r)^2}$$

$$\frac{dr}{d\lambda} = \pm\sqrt{2(\mathcal{E}-U(r))}$$

$$dr = \pm\sqrt{2(\mathcal{E}-U(r))}\, d\lambda \qquad d\theta = \frac{\ell}{R(r)^2}\, d\lambda$$

OR

$$r - r_0 = \pm\int_{\lambda_0}^{\lambda} \sqrt{2(\mathcal{E}-U(r))}\, d\lambda \longrightarrow$$

$$\theta - \theta_0 = \int_{\lambda_0}^{\lambda} \frac{\ell}{R(r(\lambda))^2}\, d\lambda$$

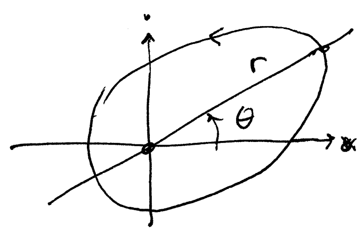
$$\frac{dr}{d\theta} = \frac{\pm\sqrt{2(\mathcal{E}-U(r))}\,\cancel{d\lambda}}{\ell/R(r)^2\,\cancel{d\lambda}} = \pm\frac{R(r)^2}{\ell}\sqrt{2(\mathcal{E}-U(r))}$$

$$\frac{d\theta}{dr} = \pm\frac{\ell}{R(r)^2\sqrt{2(\mathcal{E}-U(r))}}$$

$$\theta - \theta_0 = \pm\int_{r_0}^{r} \frac{\ell}{R(r)^2\sqrt{2(\mathcal{E}-U(r))}}\, dr \qquad \text{orbit equation}$$

$\downarrow$

invert ? $\quad r = r(\theta)$ most useful for interpretation




Kepler orbits,

GR requires elliptical functions

many roadblocks to analytic solutions and when possible:
integration needs change of variable manipulations when possible
<u>unaided</u> — Maple returns complex garbage

$\boxed{8.5}$
8?

## orthogonal coordinate surface geodesic equations

$$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cancel{2g_{12}\,dx^1dx^2}$$ ~~general~~

$$\langle u^1, u^2 \rangle = \langle \frac{dx^1}{d\lambda}, \frac{dx^2}{d\lambda} \rangle$$

$$0 = \frac{Du^i}{d\lambda} = \frac{du^i}{d\lambda} + \Gamma^i{}_{jk}u^j u^k = \frac{du^i}{d\lambda} + \frac{1}{2}g^{ii}(g_{ij,k} - g_{jk,i} + g_{ki,j})u^j u^k$$

$$0 = \frac{Du^1}{d\lambda} = \frac{du^1}{d\lambda} + \frac{1}{2g_{11}}(g_{1j,k} - g_{jk,1} + g_{k1,j})u^j u^k$$

$$= \frac{du^1}{d\lambda} + \frac{1}{2g_{11}}\left[\begin{array}{l} (g_{11,1} - \cancel{g_{11,1}} + \cancel{g_{11,1}})(u^1)^2 \\ + (\cancel{g_{12,2}} - g_{22,1} + \cancel{g_{21,2}})(u^2)^2 \\ + \left(\begin{array}{l} g_{11,2} - \cancel{g_{12,1}} + \cancel{g_{21,1}} \\ + \cancel{g_{12,1}} - \cancel{g_{21,1}} + g_{11,2} \end{array}\right) u^1 u^2 \end{array}\right]$$

$$0 = \frac{du^1}{d\lambda} + \frac{1}{2g_{11}}\left( g_{11,1}(u^1)^2 - g_{22,1}(u^2)^2 + 2g_{11,2}u^1u^2 \right)$$

$$0 = \frac{du^2}{d\lambda} + \frac{1}{2g_{22}}\left( g_{22,2}(u^2)^2 - g_{11,2}(u^1)^2 + 2g_{22,1}u^1u^2 \right)$$

metric independent of $x^2$: $\quad g_{ij,2} = 0$

$$0 = \frac{du^1}{d\lambda} + \frac{1}{2g_{11}}\left( g_{11,1}(u^1)^2 - g_{22,1}(u^2)^2 \right)$$

$$0 = \frac{du^2}{d\lambda} + \frac{1}{g_{22}}\left( g_{22,1}u^1u^2 \right) = \frac{1}{g_{22}}\frac{d}{d\lambda}(\underbrace{g_{22}u^2}_{\ell}) = \frac{1}{g_{22}}\frac{d\ell}{d\lambda}$$

$$E = g_{11}, \quad G = g_{22}, \quad F = g_{12} = 0$$

$\boxed{8.9}$

1. geodesics = autoparallel curve (local condition) = extremal distance curve between 2 points (global condition)

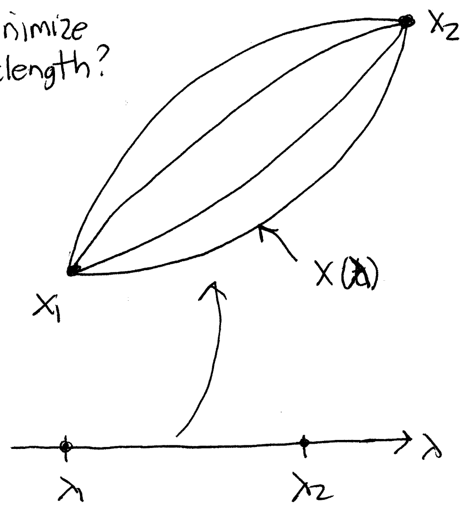
$f(x^1, \dots, x^n)$ extremized? $\hookrightarrow$ $x^{n+1} = f(x^1, \dots, x^n)$ look for horizontal tangent plane of graph

$\dfrac{\partial f}{\partial x^i} = 0$ critical point of function

(need extra tests to confirm extremal pt)

---

minimize arclength?




$\{x^i\}$ n-dimensional space with metric $g_{ij}$ — positive-definite

$x_1, x_2$ two fixed points (boundary conditions)

$x(\lambda)$ = parametrized curve from $x_1$ to $x_2$:

$x(\lambda_1) = x_1$, $x(\lambda_2) = x_2$ fixed

arclength along curve

$$\frac{ds^2}{d\lambda^2} = g_{ij}\frac{dx^i dx^j}{d\lambda\, d\lambda}$$

$$\frac{ds}{d\lambda} = \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}} = \text{length tangent vector}$$

$$s = \int_{\lambda_1}^{\lambda_2} ds = \int_{\lambda_1}^{\lambda_2} \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\, d\lambda \leftarrow ds = \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\, d\lambda$$

$$= \int_{\lambda_1}^{\lambda_2} \underbrace{\sqrt{g_{ij}(x(\lambda))\frac{dx^i(\lambda)}{d\lambda}\frac{dx^j(\lambda)}{d\lambda}}}\, d\lambda$$

$$\to L\left(x(\lambda), \frac{dx}{d\lambda}(\lambda)\right) = L(\underbrace{x^1(\lambda), \dots, x^n(\lambda), \frac{dx^1}{d\lambda}(\lambda), \dots, \frac{dx^n}{d\lambda}(\lambda)}_{\text{2n variables along curve: position, velocity}})$$

Lagrangian function

$$= \underbrace{L\left(x^1, \dots, x^n, \frac{dx^1}{d\lambda}, \dots, \frac{dx^n}{d\lambda}\right)}_{\text{function on "velocity phase space"}} \circ x(\lambda)$$

$$= \int_{\lambda_1}^{\lambda_2} L\left(x, \frac{dx}{d\lambda}\right) d\lambda$$ = functional of curve,

space of curves = $\infty$-dimensional space!

want to extremize on this space but how?

$\boxed{8.9}$
2

Narrow attention: suppose we have only a 1-parameter family of curves from $x_1$ to $x_2$:

$x^i = x^i(\lambda,\sigma)$ such that $x^i(\lambda,0) = x^i(\lambda)$ extremal curve at $\sigma=0$

Insist that arclength be a critical point as a function of $\sigma$

$$0 = \frac{d}{d\sigma}\left(\int_{\lambda_1}^{\lambda_2} ds\right)\Big|_{\sigma=0} = \frac{d}{d\sigma}\left(\int_{\lambda_1}^{\lambda_2} L\left(x(\lambda,\sigma), \frac{dx}{d\lambda}(\lambda,\sigma)\right) d\lambda\right)\Big|_{\sigma=0}$$

$$= \int_{\lambda_1}^{\lambda_2} \underbrace{\frac{d}{d\sigma} L\left(x(\lambda,\sigma), \frac{dx}{d\lambda}(\lambda,\sigma)\right)} d\lambda\Big|_{\sigma=0} \quad \begin{pmatrix}\text{since } \lambda_1,\lambda_2 \text{ ind of } \sigma \\ \text{fixed endpoints}\end{pmatrix}$$

chain rule

$$\frac{\partial L}{\partial x^i}\left(x(\lambda,\sigma), \frac{dx}{d\lambda}(\lambda,\sigma)\right)\frac{\partial x^i}{\partial\sigma}(\lambda,\sigma) + \frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)}\left(x(\lambda,\sigma), \frac{dx}{d\lambda}(\lambda,\sigma)\right)\frac{\partial}{\partial\sigma}\left(\frac{dx}{d\lambda}(\lambda,\sigma)\right)$$

$\frac{dx}{d\lambda}(\lambda) \to \frac{\partial x}{\partial\lambda}(\lambda,\sigma)$ since now depends on 2 variables so should use partial notation

$$\frac{\partial}{\partial\sigma}\left(\frac{\partial x}{\partial\lambda}(\lambda,\sigma)\right) = \frac{\partial^2 x}{\partial\sigma\,\partial\lambda}(\lambda,\sigma) = \frac{\partial}{\partial\lambda}\left(\frac{\partial x}{\partial\sigma}(\lambda,\sigma)\right)$$

commute

$$= \int_{\lambda_1}^{\lambda_2} \frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial\sigma} + \underbrace{\overbrace{\frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)}}^{f}\ \frac{\partial}{\partial\lambda}\overbrace{\left(\frac{\partial x^i}{\partial\sigma}\right)}^{g}}\ d\lambda\ \Big|_{\sigma=0}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)}\frac{\partial x^i}{\partial\sigma}\right) - \frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)}\right)\frac{\partial x^i}{\partial\sigma}$$

"integration by parts"
= product rule:

$$\begin{cases}\frac{d}{d\lambda}(f(\lambda)g(\lambda)) = \frac{df}{d\lambda}(\lambda)g(\lambda) + f(\lambda)\frac{dg}{d\lambda}(\lambda) \\ \frac{d}{d\lambda}(f(\lambda)g(\lambda)) - \frac{df}{d\lambda}(\lambda)g(\lambda) = f(\lambda)\frac{dg}{d\lambda}(\lambda)\end{cases}$$

$$= \left\{\int_{\lambda_1}^{\lambda_2}\left[\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)}\right)\right]\frac{\partial x^i}{\partial\sigma}\,d\lambda + \underbrace{\int_{\lambda_1}^{\lambda_2}\overbrace{\frac{\partial}{\partial\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)}\frac{\partial x^i}{\partial\sigma}\right)}^{\text{"total derivative"}} d\lambda}\right\}\Big|_{\sigma=0}$$

$$= \frac{\partial L}{\partial\left(\frac{dx^i}{d\lambda}\right)}\underbrace{\frac{\partial x^i}{\partial\sigma}\Big|_{\lambda_1}^{\lambda_2}}_{0} = 0$$

BUT
$$\left.\begin{aligned}x^i(\lambda_1,\sigma) &= x_1^i \to \frac{\partial x^i}{\partial\sigma}(\lambda_1,\sigma) = 0 \\ x^i(\lambda_2,\sigma) &= x_2^i \to \frac{\partial x^i}{\partial\sigma}(\lambda_2,\sigma) = 0\end{aligned}\right\}$$

constants (boundary conditions)

$\boxed{8.9}$
3

$$0 = \left\{ \int_{\lambda_1}^{\lambda_2} \left( \frac{\partial L}{\partial x^i} - \frac{\partial}{\partial \lambda} \frac{\partial L}{\partial \left(\frac{dx^i}{d\lambda}\right)} \right) \frac{\partial x^i}{\partial \sigma} \, d\lambda \right\} \Bigg|_{\sigma=0}$$

$$= \int_{\lambda_1}^{\lambda_2} \left( \frac{\partial L}{\partial x^i}(x, \tfrac{dx}{d\lambda}) - \underset{\uparrow \, \frac{d}{d\lambda}}{\frac{\partial}{\partial \lambda}} \frac{\partial L}{\partial \left(\frac{dx^i}{d\lambda}\right)}(x, \tfrac{dx}{d\lambda}) \right) \left( \frac{\partial x^i(\lambda, \sigma)}{\partial \sigma} \Big|_{\sigma=0} \right) d\lambda$$

since now $\sigma$ is gone.
function only of $\lambda$

$$= \int_{\lambda_1}^{\lambda_2} \underbrace{C_i(\lambda)}_{\text{fixed functions on extremal curve}} \, X^i(\lambda) \, d\lambda$$

can be any (differentiable!) function which vanishes at endpoints $\lambda_1, \lambda_2$:

$X^i(\lambda) = \frac{\partial x^i}{\partial \sigma}(\lambda, \sigma) \Big|_{\sigma=0} = 0$ at $\lambda_1, \lambda_2$

fixed endpoint condition

only way to guarantee that integral is zero is if $C_i(\lambda) = 0$

(pick $X^i(\lambda)$ same sign as each $C_i(\lambda)$
$\to$ positive integrand $\to$ positive integral)

Conclusion: along extremal curve must have

$$\frac{\partial L}{\partial x^i} - \frac{d}{d\lambda} \Bigg( \underbrace{\frac{\partial L}{\partial \left(\frac{dx^i}{d\lambda}\right)}}_{\equiv p_i} \Bigg) = 0$$

Lagrange derivative of Lagrangian function

$\equiv p_i$ momentum "conjugate" to variable $x^i$

$\boxed{8.9}$
4

apply to $L = \left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{1/2}$ length of tangent vector

$$\frac{\partial}{\partial x^k}\left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{1/2} = \frac{1}{2}(\ldots)^{-1/2}\underbrace{\frac{\partial}{\partial x^k}\left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)}_{g_{ij,k}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}$$

$x^k$, $\frac{dx^k}{d\lambda}$ ind vars!

$$\underbrace{\frac{\partial}{\partial\left(\frac{dx^k}{d\lambda}\right)}\left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{1/2}} = \frac{1}{2}(\ldots)^{-1/2}\underbrace{\frac{\partial}{\partial\left(\frac{dx^k}{d\lambda}\right)}\left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)}_{2g_{ij}\frac{dx^i}{d\lambda}\delta^j{}_k \,=\, 2g_{ik}\frac{dx^i}{d\lambda}}$$

$$= L^{-1} g_{ik}\frac{dx^k}{d\lambda}$$

$$\frac{d}{d\lambda}(\ ) = \frac{d}{d\lambda}\left(L^{-1}g_{ik}\frac{dx^k}{d\lambda}\right)$$

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^k}{d\lambda}\right)}\right) - \frac{\partial L}{\partial x^k} = \frac{d}{d\lambda}\left(L^{-1}g_{ik}\frac{dx^k}{d\lambda}\right) - \frac{L^{-1}}{2}g_{ij,k}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}$$

$$= L^{-1}\left(\frac{d}{d\lambda}\underbrace{\left(g_{ik}\frac{dx^i}{d\lambda}\right)}_{U_k} - \frac{1}{2}g_{ij,k}\underbrace{\frac{dx^i}{d\lambda}}_{U^i}\underbrace{\frac{dx^j}{d\lambda}}_{U^j}\right) + \left(\frac{d}{d\lambda}L^{-1}\right)\underbrace{g_{ik}\frac{dx^i}{d\lambda}}_{U_k}$$

$$\frac{dU_k}{d\lambda} - \underbrace{\frac{1}{2}g_{ij,k}}_{\Gamma_{(ij)k}}U^iU^j = \frac{DU_k}{d\lambda}$$

$$= L^{-1}\left(\frac{DU_k}{d\lambda} + \underbrace{L\frac{d}{d\lambda}L^{-1}}_{\substack{-L(L^{-2})\frac{dL}{d\lambda} \\ = -L^{-1}\frac{dL}{d\lambda} = -\frac{d\ln L}{d\lambda}}}U_k\right) = L^{-1}\left(\frac{DU_k}{d\lambda} - \left(\frac{d\ln L}{d\lambda}\right)U_k\right) = 0$$

if L constant:

$$\frac{DU_k}{d\lambda} = 0$$

$$\frac{DU^k}{d\lambda} = 0$$

affinely parametrized geodesic equations

$\boxed{8.9}$ If $L$ not constant, i.e., $U = \left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{1/2}$ = "speed" not constant

5 then length tangent vector is changing so at most can expect that its direction does not change:

$$\frac{DU^k}{d\lambda} = \frac{d\ln L}{d\lambda}U^k \propto U^k$$

same curves but more general parametrizations

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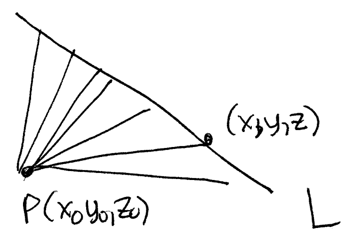
if arclength extremized, what else?

Recall:




Find shortest distance from point to line

$$d = \left((x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right)^{1/2}$$

$$d^2 = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2$$

if distance minimized so is its square. (obvious, no?)

$$0 = \frac{d}{d\sigma}\left(\int_{\lambda_1}^{\lambda_2} L\,d\lambda\right)\Big|_{\sigma=0} = \int_{\lambda_1}^{\lambda_2} \frac{dL}{d\sigma}\Big|_{\sigma=0} d\lambda$$

Consider $\frac{d}{d\sigma}\left(\int_{\lambda_1}^{\lambda_2} f(L)\,d\lambda\right)\Big|_{\sigma=0}$

$$= \int_{\lambda_1}^{\lambda_2} \frac{d}{d\sigma} f(L)\Big|_{\sigma=0} d\lambda = \int_{\lambda_1}^{\lambda_2} f'(L)\left(\frac{dL}{d\sigma}\right)\Big|_{\sigma=0} d\lambda$$

chain rule again

$$= \ldots = 0$$ also extremized

choose $\frac{1}{2}L^2$ as new Lagrangian:

$$\int_{\lambda_1}^{\lambda_2} \frac{1}{2}g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\,d\lambda = \int_{\lambda_1}^{\lambda_2}\frac{1}{2}\left(\frac{ds}{d\lambda}\right)^2 d\lambda = \int_{\lambda_1}^{\lambda_2}\frac{1}{2}\frac{ds}{d\lambda}\frac{ds}{\cancel{d\lambda}}\cancel{d\lambda}$$

$$= \int_{\lambda_1}^{\lambda_2}\frac{1}{2}\left(\frac{ds}{d\lambda}\right)ds$$

depends on parametrization unlike $\int_{\lambda_1}^{\lambda_2} ds$

8.9

6

$$L = \left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)^{1/2} \longrightarrow \frac{\partial L}{\partial x^k} = \frac{1}{2}L^{-1}\frac{\partial}{\partial x^k}\left(g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)$$

$$= (L^{-1})\frac{\partial}{\partial x^k}\underbrace{\left(\frac{1}{2}g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\right)}$$

if start instead with this Lagrangian

$$\downarrow$$

$$0 = L^{-1}\left(\frac{DU_k}{d\lambda} - \left(\frac{d\ln L}{d\lambda}\right)U_k\right)$$

↓ without L factor

↓ you lose this factor of $L^{-1}$ everywhere in derivation

$$\boxed{0 = \frac{DU_k}{d\lambda}}$$

get affinely parametrized geodesic equations

$L$ must be constant for this to hold

$$\begin{gathered} L \\ \| \\ \frac{ds}{d\lambda} \end{gathered}$$ "speed"

$$0 = \frac{d}{d\lambda}\left(\frac{ds}{d\lambda}\right) = \frac{d^2s}{d\lambda^2} \rightarrow s = a\lambda + b \quad \left(\begin{array}{c}\text{linearly}\\ \text{related}\end{array}\right)$$

requires affinely parametrized curve

easy <u>example</u> $\mathbb{R}^n$ with dot product.

$$L = \frac{1}{2}\delta_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}$$

$$\frac{\partial L}{\partial x^k} = 0, \quad \frac{\partial L}{\partial\left(\frac{dx^k}{d\lambda}\right)} = \delta_{ik}\frac{dx^i}{d\lambda}, \quad \frac{d}{d\lambda}\left(\frac{\partial L}{\partial\left(\frac{dx^k}{d\lambda}\right)}\right) = \delta_{ik}\frac{d}{d\lambda}\left(\frac{dx^i}{d\lambda}\right) = \delta_{ik}\frac{d^2x^i}{d\lambda^2} = 0$$

$$\frac{d^2x^i}{d\lambda^2} = 0 \rightarrow x^i = a^i\lambda + b^i \quad \text{linear functions of } \lambda$$

straight lines are geodesics

$\boxed{8.9}$
7

metric not positive-definite?

$$ds^2 = g_{ij}\, dx^i dx^j$$

$$\left(\frac{ds}{d\lambda}\right)^2 = \frac{ds^2}{d\lambda^2} = g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \begin{cases} >0 \\ =0 \\ <0 \end{cases}$$

BUT SIGN cannot change along geodesic in affine parametrization since constant
↳ ~~space~~ spacelike, null, timelike geodesics in Minkowski spacetime

$$\frac{ds^2}{d\lambda^2} \equiv \epsilon \underbrace{\frac{ds^2}{d\lambda^2}}_{\geq 0}$$

$\epsilon$ = $\operatorname{sgn}\left(\frac{ds}{d\lambda}\right)^2$

$s$ "interval" always real

but not necessarily "minimum" arclength

in Minkowski spacetime ⟶ maximum arclength

$$ds^2 = -dt^2 + d\vec{x}\cdot d\vec{x} = -d\tau^2 \quad \text{on timelike curves}$$

$$\left(\frac{d\tau}{d\lambda}\right)^2 = \left(\frac{dt}{d\lambda}\right)^2 - \frac{d\vec{x}}{d\lambda}\cdot\frac{d\vec{x}}{d\lambda} \geq 0$$

$$d\tau = \left(-\left(\frac{dt}{d\lambda}\right)^2 - \frac{d\vec{x}}{d\lambda}\cdot\frac{d\vec{x}}{d\lambda}\right)^{1/2} d\lambda$$

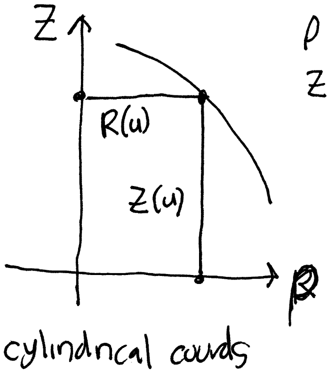
$$\text{"Action"} = \int_{\lambda_1}^{\lambda_2} d\tau = \int_{\lambda_1}^{\lambda_2}\left(-\left(\frac{dt}{d\lambda}\right)^2 - \frac{d\vec{x}}{d\lambda}\cdot\frac{d\vec{x}}{d\lambda}\right)^{1/2} d\lambda \quad + \quad ?\ \text{charge}$$

↓ $\frac{D^2x^i}{d\lambda^2} = 0$

↓ Lorentz force law

$\boxed{8.4 \text{ } \frac{8}{8.12}}$ <u>surface embedding in $\mathbb{R}^3$</u>

1




$\rho = R(u)$ parametrized
$z = Z(u)$ curve
rotated around
z axis ⟶
embedded surface
of revolution

$$ds^2 = d\rho^2 + dZ^2 + \rho^2 d\phi^2$$
$$= dR(u)^2 + dZ(u)^2 + R(u)^2 d\phi^2$$
$$= \left(R'(u)^2 + Z'(u)^2\right) du^2 + R(u)^2 d\phi^2$$

BLACK HOLE

$$= \frac{1}{\left(1-\frac{2m}{r}\right)} dr^2 + r^2 d\phi^2$$

Compare: $u = r$, $R(r) = r \longrightarrow R'(r) = 1$

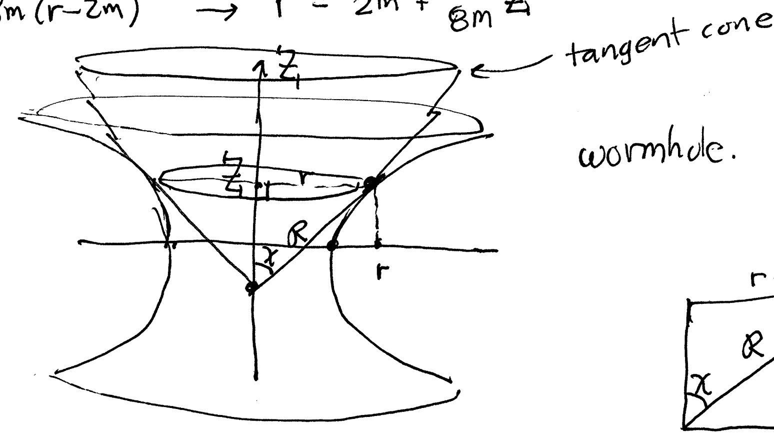
$$\underbrace{R'(r)}_{1}{}^2 + Z'(r)^2 = \frac{1}{1-\frac{2m}{r}}$$

$$Z'(r)^2 = \frac{1}{1-\frac{2m}{r}} - 1 = \frac{1}{1-\frac{2m}{r}} - \frac{1-\frac{2m}{r}}{1-\frac{2m}{r}}$$
$$= \frac{2m/r}{1-2m/r} = \frac{2m}{r-2m}$$

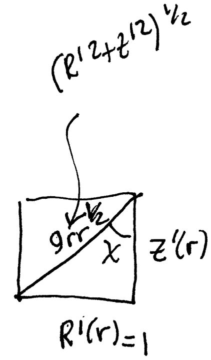
$$\frac{dZ(r)}{dr} = \sqrt{\frac{2m}{r-2m}}$$

$$Z(r) = \sqrt{8m(r-2m)}$$

$$Z(r)^2 = 8m(r-2m) \rightarrow r = 2m + \frac{1}{8m} Z^2$$

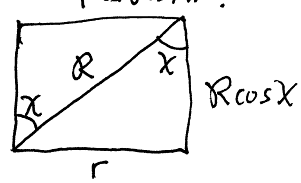



wormhole.




compare

$r = R\sin\chi$ ⟷




$$\frac{dZ}{dr} = \sqrt{8m}\ \frac{1}{2}\frac{1}{\sqrt{r-2m}} = \sqrt{\frac{2m}{r-2m}}$$
$$= \frac{R}{r}\cos\chi$$

$$\text{slope} = \frac{R}{r}\cos\chi = \cot\chi$$

$$\frac{r}{R} = \sin\chi$$
$$\shortparallel$$
$$\frac{1}{g_{rr}} = \frac{1}{\left(1-\frac{2m}{r}\right)^{-1/2}} = \left(1-\frac{2m}{r}\right)^{1/2}$$

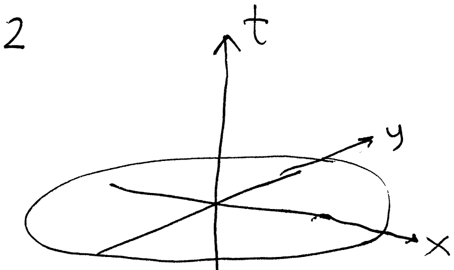
recall
<u>one loop rotation relative to ON frame (parallel transport)</u>

$$\frac{\Delta\phi}{2\pi} = 1 - \frac{r}{R} = 1 - \left(1-\frac{2m}{r}\right)^{1/2} \approx 1 - \left(1 + \frac{1}{2}\left(-\frac{2m}{r}\right) + \dots\right)$$
$$= \frac{m}{r} + \dots$$

recall:
$$R \equiv g_{rr}^{1/2}\, r \left(1-\frac{2m}{r}\right)^{-1/2}$$
$$|R'(r)|\, r\, d/r$$
$$= r\, g_{rr}^{1/2}$$

$\boxed{8.11}$ - $\boxed{8.12}$

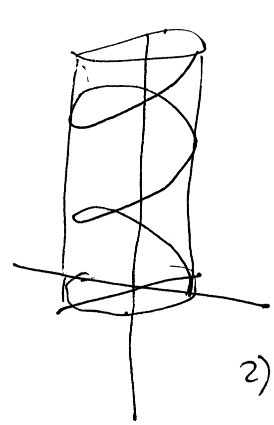
2




circle in space of $M^3$ = closed spacelike curve

we calculated parallel transport around this circle

instead



circular orbit is a helix in $M^3$ = timelike curve

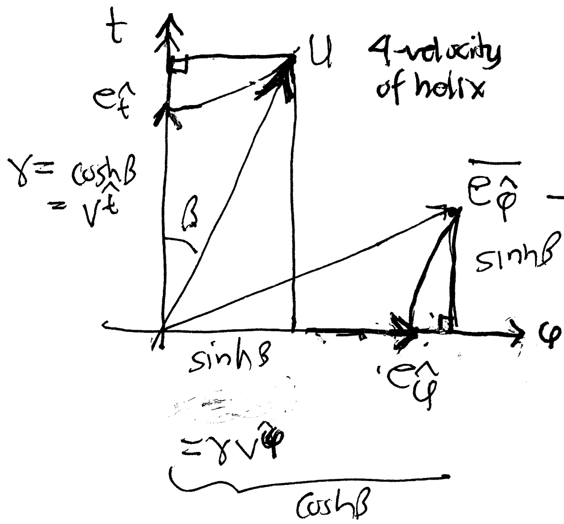
1) need to calculate 4-velocity of circular orbit (tilt of helix)

2) then constant boost in azimuthal direction of connection 1-form matrix

3) & evaluate it on 4-velocity to get rate of change

4) integrate over one loop of helix

$\boxed{\text{result gives angle change wrt polar frame after one orbit}}$




boost of normalized coord frame $e_{\hat t}, e_{\hat\varphi}$ in this plane.

azimuthal direction in local rest space of circular orbit

parallel transport along helix rotates: $e_{\hat r}, \overline{e_{\hat\varphi}} \subset LRS_u$

"gyro precession"

$$(e_{\hat t}\ e_{\hat\varphi}) \to (e_{\hat t}\ e_{\hat\varphi}) \begin{bmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{bmatrix} = (u\ \overline{e_{\hat\varphi}})$$

local time direction along helix

local azimuthal direction

$\boxed{\S 9.4 - 9.12}$ $\quad ds^2 = -(1-\frac{2m}{r})dt^2 + (1-2m/r)^{-1}dr^2 + r^2 d\theta^2 = -d\tau^2$

3

$$L = \frac{1}{2}\frac{ds^2}{d\lambda} = -\frac{1}{2}(1-\frac{2m}{r})\left(\frac{dt}{d\lambda}\right)^2 + (1-2m/r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\theta}{d\lambda}\right)^2 = -\frac{1}{2}\underbrace{\left(\frac{d\tau}{d\lambda}\right)^2}_{\mu^2 \text{ affine}}$$

$\mu^2$ affine $=$ 1 for $\lambda = \tau$ timelike

but $= 0$ for null geo

2 conserved momenta

$$p_t = \frac{\partial L}{\partial(\partial t/d\lambda)} = -(1-\frac{2m}{r})\frac{dt}{d\lambda} \equiv -\mathcal{E}$$

$$p_\theta = \frac{\partial L}{\partial(\partial t/\partial\lambda)} = r^2\frac{d\theta}{d\lambda} \equiv \ell$$

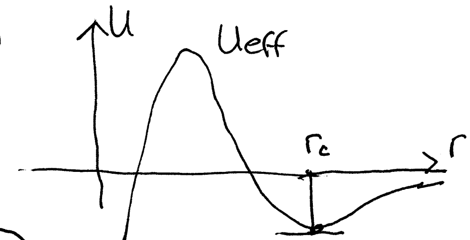
$$L = -\frac{1}{2}(1-\frac{2m}{r})^{-1}\mathcal{E}^2 + \frac{1}{2}(1-\frac{2m}{r})^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{2r^2} = -\frac{1}{2}\mu^2 \quad = \text{kinetic energy}$$

manipulate

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\frac{\ell^2}{r^2}(1-\frac{2m}{r}) - \frac{1}{2}\mathcal{E}^2 = -\frac{1}{2}\mu^2(1-\frac{2m}{r})$$

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \underbrace{\frac{1}{2}(\mu^2+\frac{\ell^2}{r^2})(1-\frac{2m}{r})}_{U_{eff}} = +\frac{1}{2}\mathcal{E}^2$$

$U_{eff}$ for radial motion.




circular orbit

$$0 = \frac{dU_{eff}}{dr} = \frac{1}{2}\left(-\frac{2\ell^2}{r^3}\right)(1-\frac{2m}{r}) + \frac{1}{2}(\mu^2+\frac{\ell^2}{r^2})\left(+\frac{2m}{r^2}\right)$$

for given $\ell$ critical point of $U_{eff}$

$$\left[ = -\frac{\ell^2}{r^3}(1-\frac{2m}{r}) + \frac{m}{r^2}(\mu^2+\frac{\ell^2}{r^2}) \right] r^4$$

$$-\ell^2(r-2m) + m(\mu^2 r^2 + \ell^2) = 0$$

$$m\mu^2 r^2 - \ell^2(r-3m) = 0$$

$$\ell^2 = \frac{\mu^2 m r^2}{(r-3m)} \rightarrow \ell = \pm \mu r\sqrt{\frac{m}{r-3m}}$$

$$\frac{d\theta}{d\lambda} = \frac{\ell}{r^2} = \pm\mu\sqrt{\frac{m}{r^3(1-3m/r)}}$$

Newtonian force balance (circular orbit)

$$\frac{m}{r^2} = \frac{v^2}{r}$$

g-acc $\quad$ centripetal acc.

$$v = \pm\sqrt{\frac{m}{r}}$$

$$\omega = \frac{v}{r} = \pm\sqrt{\frac{m}{r^3}}$$

$\omega = d\theta/d\tau \quad \uparrow \mu = 1$

$$\gamma v^{\hat\theta} = r\frac{d\theta}{d\tau} = \pm\sqrt{\frac{m}{r(1-3m/r)}} = \sinh\beta$$

$\frac{m}{r} \ll 1$ : Newtonian limit!

stable circular orbits only for $r > 3m$

$\boxed{8.4-8.12}$

3b

$$\gamma = \cosh\beta = \sqrt{1+\sinh^2\beta} = \sqrt{1+\frac{m}{r-3m}} = \sqrt{\frac{r-3m+m}{r-3m}} = \sqrt{\frac{r-2m}{r-3m}}$$

$$\gamma v^{\hat\theta} = \pm\sqrt{\frac{m}{r-3m}} = \sinh\beta$$

$$v^{\hat\theta} = \frac{\gamma v^{\hat\theta}}{\gamma} = \pm\sqrt{\frac{m}{r-3m}}\sqrt{\frac{r-3m}{r-2m}} = \pm\sqrt{\frac{m}{r-2m}} = \tanh\beta.$$

$$u = \gamma e_{\hat t} + \gamma v^{\hat\theta} e_{\hat\theta} = \underbrace{\sqrt{\frac{r-2m}{r-3m}}}_{\cosh\beta} e_{\hat t} + \underbrace{\sqrt{\frac{m}{r-3m}}}_{\sinh\beta} e_{\hat\theta}$$

counterclockwise geo.

$$\overline{e_{\hat\theta}} = \gamma e_{\hat\theta} + \gamma v^{\hat\theta} e_{\hat t} = \sinh\beta\, e_{\hat t} + \cosh\beta\, e_{\phi}$$
$$= \sqrt{\frac{m}{r-3m}}\, e_{\hat t} + \sqrt{\frac{r-2m}{r-3m}}\, e_{\phi}$$

notice $v^{\hat\theta} = \pm\sqrt{\frac{m}{r-2m}}$ is valid up to $r=2m$ but these values do not come from the critical point condition which requires at least a horizontal tangent in the graph of the effective potential.

indeed as one lowers the angular momentum magnitude these critical points disappear at $r=3m$

so these correspond to... need more expert discussion here.

$\boxed{8.4-8.12}$

4

$$e_{\hat t} = \sqrt{1-\tfrac{2m}{r}}\,\partial_t \;\longrightarrow\; \partial_t = \left(1-\tfrac{2m}{r}\right)^{-1/2} e_{\hat t}$$
$$e_{\hat r} = \left(1-\tfrac{2m}{r}\right)^{-1/2}\partial_r \;\longrightarrow\; \partial_r = \left(1-\tfrac{2m}{r}\right)^{1/2} e_{\hat r}$$
$$e_{\hat\theta} = r^{-1}\partial_\theta \;\longrightarrow\; \partial_\theta = r\, e_{\hat\theta}$$

to re-express coord frame in terms of O.N. frame

$$\underbrace{\phantom{xx}}_{}\; C^{\hat t}{}_{\hat r \hat t}$$

$$[e_{\hat r}, e_{\hat t}] = \left(1-\tfrac{2m}{r}\right)^{-1/2} \underbrace{\partial_r\left(1-\tfrac{2m}{r}\right)^{1/2}}\,\partial_t = \frac{m}{r^2}\left(1-\tfrac{2m}{r}\right)^{-1}\partial_t = \frac{m}{r^2}\left(1-\tfrac{2m}{r}\right)^{-1/2} e_{\hat t}$$

$$[e_{\hat r}, e_{\hat\theta}] = \left(1-\tfrac{2m}{r}\right)^{-1/2} \underbrace{\partial_r(r^{-1})}\,\partial_\theta = -\frac{1}{r^2}\left(1-\tfrac{2m}{r}\right)^{-1/2}\partial_\theta = \underbrace{-\frac{1}{r}\left(1-\tfrac{2m}{r}\right)^{-1/2}}_{C^{\hat\theta}{}_{\hat r\hat\theta}} e_{\hat\theta}$$

<u>need 2 $\hat\theta$ indices or 2 $\hat t$ indices for nonzero connection component</u>

$$\Gamma_{\hat r\hat\theta\hat\theta} = \tfrac12\left(\cancel{C_{\hat r\hat\theta\hat\theta}}^{\,0} - \underset{0}{C_{\hat\theta\hat\theta\hat r}} + C_{\hat\theta\hat r\hat\theta}\right) = C_{\hat\theta\hat r\hat\theta} \to \Gamma^{\hat r}{}_{\hat\theta\hat\theta} = C^{\hat\theta}{}_{\hat r\hat\theta} = -K(\theta)$$

$$\Gamma_{\hat r\hat t\hat t} = \tfrac12\left(\cancel{C_{\hat r\hat t\hat t}}^{\,0} - C_{\hat t\hat t\hat r} + C_{\hat t\hat r\hat t}\right) = C_{\hat t\hat r\hat t} \to \Gamma^{\hat r}{}_{\hat t\hat t} = C^{\hat t}{}_{\hat r\hat t} = K(t)$$

$$\hat\omega = \begin{array}{c|ccc} & t & r & \theta \\ \hline t & 0 & C^{\hat t}{}_{\hat r\hat t} & 0 \\ r & \pm C^{\hat t}{}_{\hat r\hat t} & 0 & C^{\hat\theta}{}_{\hat r\hat\theta} \\ \theta & 0 & -C^{\hat\theta}{}_{\hat r\hat\theta} & 0 \end{array} = \begin{bmatrix} 0 & K(t) & 0 \\ +K(t) & 0 & K(\theta) \\ \cancel{K} & -K(\theta) & 0 \end{bmatrix}$$

'' $\Gamma^{\hat t}{}_{\hat t\hat r} = -\Gamma_{\hat t\hat t\hat r} = \Gamma_{\hat r\hat t\hat t}$ index raised changes sign antisym lower $= \Gamma^{\hat r}{}_{\hat t\hat t}$

'' acceleration of t lines

antisym when index lowered

[upper 2x2 block symmetric
lower 2x2 block antisymmetric]

$$K(\theta) = \frac{1}{r(1-2m/r)^{1/2}} \qquad R(\theta) = r\sqrt{1-\tfrac{2m}{r}} \to \nabla_{e_{\hat\theta}} e_{\hat\theta} = K(\theta)\underbrace{(-e_{\hat r})}_{\hat N}$$

inward radial normal

$$K(t) = \frac{m}{r^2\sqrt{1-2m/r}} = \text{outward acceleration} \to \nabla_{e_{\hat t}} e_{\hat t} = K(t)\, e_{\hat r}$$

relativistic corrections when $\frac{r}{2m} \sim 1$

last step

$$\underbrace{B\,\hat\omega(u)\,B^{-1}}$$

$B =$ constant boost to $(u, e_{\hat\theta})$ from $(e_{\hat t}, e_{\hat\theta})$

describes parallel transport of angular direction in rest frame

...

$\boxed{9.1}$ **curvature?** First the tensor, then the interpretation

1

$\mathbb{R}^n$, cartesian coords: $\{x^i\}$

$\nabla_i Z^k = \partial_i Z^k$

$$\nabla_j \nabla_i Z^k = \partial_j \partial_i Z^k = \partial_i \partial_j Z^k = \nabla_i \nabla_j Z^k$$

$$(\nabla_j \nabla_i - \nabla_i \nabla_j) Z^k = 0$$

$$[\nabla_j, \nabla_i] Z^k = 0 \qquad Z^k{}_{;ij} - Z^k{}_{;ji} = 2 Z^k{}_{;[ij]} = 0$$

commutator of covariant derivatives vanishes

what about

$$[\nabla_X, \nabla_Y] Z^k = \underbrace{[X^i \nabla_i, Y^j \nabla_j]}\, Z^k \,?$$

$$\left.\begin{aligned} &= [X^i \partial_i, Y^j \partial_j] Z^k \\ &= [X, Y] Z^k \\ &= \nabla_{[X,Y]} Z^k \end{aligned}\right\} \text{ in cartesian coords.}$$

$$\therefore \quad \underbrace{\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right)}_{\text{coordinate independent!}} Z^k = 0$$

$\nabla_Y Z^i = Z^i{}_{;j} Y^j = Z^i{}_{,j} Y^j$ cartesian coords

check in detail

$$[\nabla_X \nabla_Y - \nabla_Y \nabla_X] Z^i = [\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z)]^i$$

$$= (\nabla_Y Z)^i{}_{,k} X^k - (\nabla_X Z)^i{}_{,k} Y^k$$

$$= (Z^i{}_{,j} Y^j)_{,k} X^k - (Z^i{}_{,j} X^j)_{,k} Y^k$$

$$= Z^i{}_{,jk} Y^j X^k - Z^i{}_{,jk} X^j Y^k \quad \text{switch dummy indices}$$

$$+ Z^i{}_{,j} Y^j{}_{,k} X^k - Z^i{}_{,j} X^j{}_{,k} Y^k$$

$$= \underset{\overset{\shortparallel}{0}}{[Z^i{}_{,jk} - Z^i{}_{,kj}]} X^k Y^j + \underbrace{Z^i{}_{,j} \underbrace{(X^k Y^j{}_{,k} - Y^k X^j{}_{,k})}_{[X,Y]^j}}_{\nabla_{[X,Y]} Z^i}$$

$$\therefore \; (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z^i = 0$$

$\boxed{9.1}$

1b

HISTORICALLY reverse calculation

$$\nabla_i X^k = \partial_i X^k + \Gamma^k{}_{im} X^m$$

$$\partial_i X^k = \nabla_i X^k - \Gamma^k{}_{im} X^m$$

$$\partial_j \partial_i X^k = \nabla_j (\nabla_i X^k - \Gamma^k{}_{im} X^m) - \Gamma^k{}_{jn} (\nabla_i X^n - \Gamma^n{}_{im} X^m)$$

$$\partial_i \partial_j X^k = \dots$$

$$0 = (\partial_j \partial_i - \partial_i \partial_j) X^k = \dots$$

integrability condition for PDEs for cov constant vector field

$\boxed{9.1}$
2

Now evaluate in general coords:

$$\nabla_Y Z^i = Z^i{}_{;j} Y^j = (Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m) Y^j$$

$$[\nabla_X \nabla_Y Z]^i = ([\nabla_Y Z]^i{}_{,k} + \Gamma^i{}_{km} [\nabla_Y Z]^m) X^k$$

$$= \{ \underbrace{[(Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m) Y^j]_{,k}}_{\text{expand}} + \underbrace{\Gamma^i{}_{km}(Z^m{}_{,j} + \Gamma^m{}_{jp} Z^p) Y^j \} X^k}_{\text{no change}}$$

$$= \{ (\underset{①}{Z^i{}_{,jk}} + \underset{②}{\Gamma^i{}_{jm} Z^m{}_{,k}} + \underset{③}{\Gamma^i{}_{jm,k}}) Y^j + \underbrace{(Z^i{}_{,j} + \Gamma^i{}_{jm} Z^m)}_{Z^i{}_{;j}} \underset{④}{} Y^j{}_{,k}$$

$$+ \Gamma^i{}_{km}(\underset{⑤}{Z^m{}_{,j}} + \underset{⑥}{\Gamma^m{}_{jp} Z^p}) Y^j \} X^k$$

$$= [\underset{①}{Z^i{}_{,jk}} + \underset{②}{\Gamma^i{}_{jm} Z^m{}_{,k}} + \underset{⑤}{\Gamma^i{}_{km} Z^m{}_{,j}} + \underset{③}{\Gamma^i{}_{jm,k}} + \underset{⑥}{\Gamma^i{}_{km}\Gamma^m{}_{jp} Z^p}] X^k Y^j$$

$$+ \underset{④}{Z^i{}_{;j} Y^j{}_{,k} X^k}$$

if we switch X, Y and subtract → antisymmetrize

symmetric terms in (jk) cancel out.

① , ②+⑤ → 0

upper line contribution

$$[\nabla_X \nabla_Y Z]^i = [\ldots]^i{}_{jk} X^k Y^j \quad \text{relable}$$

$$-[\nabla_Y \nabla_X Z]^i = -[\ldots]^i{}_{jk} X^j Y^k = -[\ ]^i{}_{kj} X^k Y^j$$

$$[\nabla_X \nabla_Y - \nabla_Y \nabla_X] Z^i = ([\ldots]^i{}_{jk} - \underbrace{[\ldots]^i{}_{kj}}) X^k Y^j$$

symmetric terms cancel

only ③+⑥ remain in difference — antisym in (j,k)

lower line contribution

$$\underbrace{Z^i{}_{;j} \underbrace{(X^k Y^j{}_{,k} - Y^k X^j{}_{,k})}_{[X,Y]^j}}_{(\nabla_{[X,Y]} Z)^i}$$

$$(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}) Z^i$$

$$= (\Gamma^i{}_{jm,k} - \Gamma^i{}_{km,j} + \Gamma^i{}_{km}\Gamma^m{}_{jp} - \Gamma^i{}_{jm}\Gamma^m{}_{kp}) \underbrace{X^k Y^j Z^m}$$

no derivatives !! multilinear function

tensor!! $R^i{}_{mkj}$

$\boxed{9.1}$ 3

$\mathbb{R}^n$, any coords, $R^i{}_{jkl} \equiv 0$

$$\underbrace{R^i{}_j}_{\text{linear trans}}\ \underbrace{{}_{mn}}_{\text{2-form}} = \underbrace{\partial_m \Gamma^i{}_{nj} - \partial_n \Gamma^i{}_{mj}}_{\text{2nd derivatives of } g_{ij}} + \underbrace{\Gamma^i{}_{mk}\Gamma^k{}_{nj} - \Gamma^i{}_{nk}\Gamma^k{}_{mj}}_{\text{+ first derivatives of } g_{ij}}$$

this is a nightmare BUT zero for the flat space flat connection

Yes, this is our measure of curvature.

Remember calc I: $\frac{d^2}{dx^2} f(x) \to$ concavity $\to$ curvature

second derivatives

smells right.

later we interpret this in terms of parallel transport.

$$R = R^i{}_{jmn}\, \underbrace{e_i \otimes \omega^j}_{\text{linear transformation}} \otimes \omega^m \otimes \omega^n$$

only antisymmetric part contributes

$$= \tfrac{1}{2} R^i{}_{jmn}\, e_i \otimes \omega^j \otimes \underbrace{(\omega^m \wedge \omega^n)}_{\text{2-form}}$$

partial evaluation on lower 3 indices $\to R(X,Y)Z = \tfrac{1}{2} R^i{}_{jmn}\, e_i\, \omega^j(Z)\, \omega^m(X)\, \omega^n(Y)$

$(X,Y)$: evaluate 2-form factor $\to$ picks out plane of $X \wedge Y$

$Z$: undergoes linear transformation

"linear transformation valued 2-form"

measures curvature in any space with metric $g_{ij}$ giving rise to metric connection $\nabla$ with components $\Gamma^i{}_{jk}$ in any coord system

$\boxed{9.1}$
4

Frames often more useful for interpretation than coordinates

Redo calculation in general frame $\{e_i\}$, dual frame $\{\omega^i\}$

$$R^i{}_{jmn} = R(\omega^i, \underset{Z}{e_j}, \underset{X,\,Y}{e_m, e_n})$$

definition of frame components

$$\left(\nabla_{e_m}\nabla_{e_n} - \nabla_{e_n}\nabla_{e_m} - \nabla_{[e_m,e_n]}\right) e_j = R^i{}_{jmn}\, e_i$$

$$\nabla_{e_n} e_j = \Gamma^k{}_{nj}\, e_k$$

$$\nabla_{e_m}\nabla_{e_n} e_j = \nabla_{e_m}(\Gamma^k{}_{nj} e_k) = \underbrace{(\nabla_{e_m}\Gamma^k{}_{nj})}_{\Gamma^k{}_{nj,m}} e_k + \Gamma^k{}_{nj}\underbrace{\nabla_{e_m} e_k}_{\Gamma^l{}_{mk} e_l}$$

switch & subtract

switch

$\Gamma^l{}_{nj}\Gamma^k{}_{ml}\, e_k$

$$= \left(\Gamma^k{}_{nj,m} + \Gamma^k{}_{ml}\Gamma^l{}_{nj}\right) e_k$$

$$(\nabla_{e_m}\nabla_{e_n} - \nabla_{e_n}\nabla_{e_m}) e_j = \begin{pmatrix} \Gamma^k{}_{nj,m} + \Gamma^k{}_{ml}\Gamma^l{}_{nj} \\ - \Gamma^k{}_{mj,n} - \Gamma^k{}_{ml}\Gamma^l{}_{mj} \end{pmatrix} e_k$$

$$\underbrace{-\nabla_{[e_m,e_n]} e_j}\; = -\left(\nabla_{C^k{}_{mn} e_k}\right) e_j = -C^k{}_{mn}\underbrace{\nabla_{e_k} e_j}_{\Gamma^l{}_{kj} e_l}$$

switch $k, l$

$$R^k{}_{jmn}\, e_k = \left(\Gamma^k{}_{nj,m} - \Gamma^k{}_{mj,n} + \Gamma^k{}_{ml}\Gamma^l{}_{nj} - \Gamma^k{}_{nl}\Gamma^l{}_{mj} - C^l{}_{mn}\Gamma^k{}_{lj}\right) e_k$$

just one extra term
zero for coordinate frame
$[\partial_i, \partial_j] = 0$

$\boxed{9.1}$

5

$R_{ijmn} = g_{ik} R^k{}_{jmn}$ fully covariant curvature tensor

antisymmetric in $2^{nd}$ pair of indices $= -R_{ijnm}$

(2-form definition)

antisymmetric in first pair of indices

(parallel transport leads to rotation → generator of rotation

antisymmetric matrix when index lowered.

$= -R_{jimn}$

$= R_{mnij}$

symmetric in pair interchange

(like a bilinear function on 2-vectors)

$R_{mnij} S^{mn} T^{ij} = R_{mnij} T^{mn} S^{ij}$

how to prove? — manipulate formula with index lowered (see text)

how to prove? $g_{ij;m} = 0 \rightarrow g_{ij;mn} = 0$

$\rightarrow g_{ij;mn} - g_{ij;nm} = 0$

$[\nabla_m, \nabla_n] g_{ij} = 0$

$\downarrow \cdots$

$R_{[mn]ij} = 0$

$\boxed{9.1}$
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## another symmetry

recall Jacobi identity $\swarrow [X,Y] = XY - YX$

$$\underbrace{[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]}_{\text{cyclic permute}} = \dots (6 \text{ terms}) = 0$$

cancel in pairs.

recall symmetry of connection

$\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ in coords

$\nabla_X Y - \nabla_Y X = \dots = [X,Y]$ connection components cancel.

$\nabla_Z X - \nabla_X Z = [Z,X]$

$X \to [X,Y]$

$\nabla_Z \underbrace{[X,Y]}_{\nabla_X Y - \nabla_Y X} - \nabla_{[X,Y]} Z = [Z,[X,Y]]$

$$\nabla_Z \nabla_X Y - \nabla_Z \nabla_Y X - \nabla_{[X,Y]} Z = [Z,[X,Y]]$$

$$\nabla_X \nabla_Y Z - \nabla_X \nabla_Z Y - \nabla_{[Y,Z]} X = [X,[Y,Z]]$$

$$\nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Z,X]} Y = [Y,[Z,X]]$$

cyclic permute

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sum

$$R(Z,X)Y + R(X,Y)Z + R(Y,Z)X = 0 \quad \text{Jacobi identity}$$

$$\underbrace{R^m{}_{ijk} + R^m{}_{jki} + R^m{}_{kij}}_{\text{cygdic premute}} = 0 \leftarrow \text{"Blanchi identity"}$$

(consequence of symmetric connection)

$$= \frac{1}{2}\begin{pmatrix} R^m{}_{ijk} + R^m{}_{jki} + R^m{}_{kij} \\ + R^m{}_{ikj} - R^m{}_{jik} - R^m{}_{kji} \end{pmatrix}$$

$$= \frac{1}{2}\, 3!\, R^m{}_{\underbrace{[ijk]}_{\text{antisymmetric part}}}$$

$\boxed{9.1}$

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how many independent components? <u>n dimensions</u>

$R_{ij\,mn}$

$$N = \frac{n(n-1)}{2}$$

$N \times N$ sym matrix

2-form = antisym matrix

$n$

$$\begin{array}{l} n-1 \\ +\, n-2 \\ +\, \cdots \\ +\, 1 \\ +\, 0 \end{array} \qquad \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2}$$

$N$

$N$

$$\begin{array}{l} N \\ N-1 \\ \vdots \\ +1 \end{array} \qquad \frac{N(N+1)}{2}$$

symmetric linear function of bi-vectors

$$\frac{\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2}+1\right)}{2} = \frac{1}{8}\, n(n-1)(n^2-n+2)$$

$R^m{}_{ijk} = 0$ ← how many conditions?

must subtract this number

at midnight I could not figure this out

Final number (google)

| $n$ | $\frac{1}{12} n^2(n^2-1)$ | without Bianchi |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 1 | |
| 3 | 6 | 6 |
| 4 | 20 | 21 ← 1 extra condition |

} Bianchi nothing extra. consequence of other symmetries

Exercise: find explanation of this number

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$R^m{}_{ijk} \rightarrow R^m{}_{imk} \equiv R_{ik} = R_{ki}$ symmetric Ricci

1 3

$R^{mi}{}_{mi} = R^i{}_i \equiv R$ scalar curvature

$G_{ij} \equiv R_{ij} - \frac{1}{2} R\, g_{ij}$ Einstein tensor

$\boxed{9.1}$
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$\boxed{n=3:}$

$0 = 3R_{1[123]} = \cancel{R_{1123}} + R_{1231} + \underbrace{R_{1312}}_{R_{1213}} \rightarrow R_{12[31]} = 0$ not new

$\cancel{R_{1231}}$

0

$\boxed{n=4:}$

$0 = 3R_{4[123]} = R_{4123} + R_{4231} + R_{4312}$

$0 = 3R_{4[423]} = \cancel{R_{4423}} + R_{4234} + \underbrace{R_{4342}}_{R_{4243}} \Rightarrow R_{42[34]} = 0$ not new

if repeated

al ~~l others have at least one repeated~~.

$0 = 3R_{1[234]} = R_{1234} + R_{1342} + R_{1423}$

$R_{3412} + R_{4213} - R_{4123}$

four choices for index leftout.

$- R_{4312} - R_{4231} - R_{4123}$

same as first.

all collapse to 1 independent constraint.

$\boxed{n=2}$ $R_{2[\underbrace{122}]}$

3-forms identically zero

how to analyze in general?