$\boxed{7.1\text{–}7.2}$ covariant derivative stuff

1 $T^{i\dots}{}_{j\dots;k} = T^{i\dots}{}_{j\dots,k} + \Gamma^i{}_{k\ell}T^{\ell\dots}{}_{j\dots} + \dots - \Gamma^\ell{}_{kj}T^{i\dots}{}_{\ell\dots} - \dots$

$$[\nabla_X T]^{i\dots}{}_{j\dots} = T^{i\dots}{}_{j\dots;k}X^k = T^{i\dots}{}_{j\dots,k}X^k + \Gamma^i{}_{k\ell}X^k T^{\ell\dots}{}_{j\dots} + \dots - \Gamma^\ell{}_{kj}X^k T^{i\dots}{}_{\ell\dots}$$
$$= dT^{i\dots}{}_{j\dots}(X) + \omega^i{}_\ell(X)T^{\ell\dots}{}_{j\dots} + \dots - \omega^\ell{}_j(X)T^{i\dots}{}_{\ell\dots}$$

basis: $\{e_i\} \to \omega^i{}_j = \Gamma^i{}_{kj}\omega^k$ connection 1-forms, $[e_i, e_j] = C^k{}_{ij}e_k$

metric: g_{ij} : $\nabla g = 0$, ∇ "symmetric" $\longrightarrow$ $\Gamma^i{}_{jk} = \frac{1}{2}g^{i\ell}(g_{\ell j,k} - g_{jk,\ell} + g_{k\ell,j})$ same order, then ←

$+\frac{1}{2}g^{i\ell}(C_{\ell jk} - C_{jk\ell} + C_{k\ell j})$

(I never remember the index order!)

$\mathbb{R}^n$: gradient divergence — $\mathbb{R}^3$: curl differential

multivariable calc ops! what are they?

① $df = \frac{\partial f}{\partial x^i}dx^i = (e_i f)\,\omega^i \equiv f_{,i}\,\omega^i$ (coord frame/general frame)

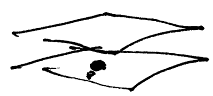
1-form = covector field, no metric needed.

- $\vec{\nabla} f = g^{ij}f_{,i}\,e_j \equiv \text{grad} f$ components $[\vec{\nabla} f]^i = g^{ij}f_{,i}$

vector field $(df)^\# = \vec{\nabla} f$ need metric to raise index to get vector field from 1-form

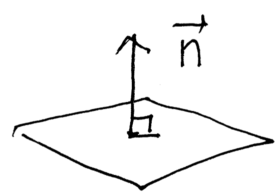
$\mathbb{R}^3$ Cartesian coords: $(\vec{\nabla} f)^i = \delta^{ij}f_{,i}$ no need to distinguish vector field / 1-form

interpretation very different: df linearizes level surface family to tangent space



linear approximation for nearby points ~ vectors in tangent space

$\vec{\nabla} f$ is a normal to tangent plane to level surface



- directional derivative

$df(X) = Xf = \nabla_X f = X^i f_{,i}$ (no metric needed) linear approximation evaluation

$= X \cdot \vec{\nabla} f = g_{ij}X^i\nabla^j f$ (metric required) "~~det~~ dot del"

$\mathbb{R}^3$ Cartesian coordinates, vector field $X = X^i \partial_i$, $X^\flat = \delta_{ij} X^i dx^j$

$$\mathrm{curl}\, X = \epsilon^{kij} \partial_i X_j \partial_k = \eta^{kij} \partial_i X_j \partial_k = \eta^{kij} X_{j,i} \partial_k$$

$$\boxed{\eta^{kij} = (\det g)^{-1/2} \epsilon^{kij}}$$
in oriented frame

If start with 1-form $X_i dx^i$, no metric required.
To start with vector field $X^i \partial_i$, need metric to lower index

$$\mathrm{curl}\, X = \eta^{kij} \partial_i X_j \partial_k = \eta^{kij} X_{j,i} \partial_k$$

$$= \eta^{kij} (X_{j;i} - \Gamma^{\ell}{}_{ij} X_\ell) \partial_k = \eta^{kij} X_{j;i} \partial_k$$

$$\Rightarrow \eta^{kij} (g_{jm} X^m)_{;i} = \eta^{kij} g_{jm} X^m{}_{;i}$$

$$= \eta^{kij} (\nabla X^\flat)^\sharp{}_{ji}$$

$$= \underbrace{g^{kn} \underbrace{\eta_n{}^{ij} (\nabla X^\flat)_{ji}}_{(\nabla \wedge X^\flat)^*{}_n}}_{[((\nabla \wedge X^\flat)^*)^\sharp]^k}$$

$$\mathrm{curl}\, X = \underbrace{((\nabla \wedge X^\flat)^*)^\sharp}_{\text{valid in any frame}}$$

1, 2, 3, 4, 5

5 operations in succession
only for $n=3$
dual of 2-form = 1-form
$\updownarrow$
vector field

Chapter 11: $(\mathrm{curl}\, X^\flat)^\flat = dX^\flat$ exterior derivative of 1-form
no metric needed.
If start with 1-form

simple:
valid for all n

no metric, no connection

cross product in $\mathbb{R}^3$

$$[X \times Y]^i = \epsilon_{ijk} X^j Y^k = \delta^{il} \epsilon_{ljk} X^j Y^k \quad \text{Cartesian coords.}$$

$$= \eta^i{}_{jk} X^j Y^k \quad \text{any coords / frame.}$$

- $$\text{curl}\, X = \eta^{kij} \nabla_i X_j$$

$$= \vec{\nabla} \times X \qquad \text{"del cross"}$$

"del dot" = divergence

- $$\text{div}\, X = \frac{\partial X^i}{\partial x^i} = X^i{}_{,i} = \underbrace{X^i{}_{;i}}_{\text{valid in any frame}} \quad \text{in Cartesian coords}$$

$$= \vec{\nabla} \cdot X$$

$$= X^i{}_{,i} + \Gamma^i{}_{ik} X^k$$

$$\Gamma^i{}_{jk} = \tfrac{1}{2} g^{il}(g_{lj,k} - g_{jk,l} + g_{kl,j}) + \tfrac{1}{2}(C^i{}_{jk} - C_{jk}{}^i + C_k{}^i{}_j)$$

$$\Gamma^i{}_{ik} = \tfrac{1}{2} \underbrace{g^{il}}_{\text{sym}} (g_{li,k} \underbrace{- g_{ik,l} + g_{kl,i}}_{\text{antisym} \to 0}) + \tfrac{1}{2}(\underbrace{C^i{}_{ik} - C_{ik}{}^i}_{C^i{}_{ik} = -C^i{}_{ki}} + C_k{}^i{}_i)$$

$$- g_{ikl} + g_{lk,i} \qquad + C^i{}_{ik} \qquad C_{kli} g^{li} \to 0$$

$$= \tfrac{1}{2} \mathrm{Tr}(\underline{g}^{-1} \underline{g}_{,k})$$

$$\tfrac{1}{2} \underbrace{\mathrm{Tr}(\underline{g}^{-1} d\underline{g})}_{d \ln|\det(\underline{g})|} (\partial_k) = d(\ln|\det \underline{g}|^{1/2})$$

$$\Gamma^i{}_{ik} = (\ln|\det \underline{g}|^{1/2})_{,k} - \underbrace{C^i{}_{ki}}_{= 0 \text{ in coord frame}}$$

$\boxed{7.1-7.2}$

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$$\mathrm{div}\,\mathbf{X} = X^i{}_{,i} + \Gamma^i{}_{ik} X^k$$

$$= \underbrace{X^i{}_{;i}}_{\frac{|\det g|^{1/2} X^i{}_{,i}}{|\det g|^{1/2}}} + \underbrace{(\ln|\det g|^{1/2})_{,k} X^k}_{\frac{(|\det g|^{1/2})_{,k} X^k}{|\det g|^{1/2}}} - C^i{}_{ki} X^k$$

$$\boxed{\mathrm{div}\,\mathbf{X} = \frac{(|\det g|^{1/2} X^i)_{,i}}{|\det g|^{1/2}} - \underbrace{C^i{}_{ki} X^k}_{}}$$

= 0 in coord frame

recall $\eta_{i_1 \dots i_n} = |\det g|^{1/2} \epsilon_{i_1 \dots i_n}$ volume n-form

divergence only needs an n-form from metric

[Chapter 11 : exterior derivative of (n−1) form — no metric required.]

simpler to evaluate in coord system (especially an orthogonal one)

$\boxed{7.1-7.2}$
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second derivatives

$$T^i{}_{j;kl} \equiv T^i{}_{j;k;l} \equiv [\nabla\nabla T]^i{}_{jkl} = [\nabla_l \nabla_k T]^i{}_j$$

derivative indices add on to the right

del indices add on to left

same abbreviation for partial derivatives, frame derivatives

$$T^i{}_{j,k,l} \equiv T^i{}_{j,kl} = \partial_l \partial_k T^i{}_j = e_l e_k T^i{}_j \quad \text{(frame)}$$

function:

$$\nabla f = f_{;i}\,\omega^i$$

$$\nabla\nabla f = f_{;ij}\,\omega^i \otimes \omega^j$$

$$f_{;ij} = (f_{,i})_{,j} - f_{,k}\Gamma^k{}_{ij} = f_{,ij} - f_{,k}\Gamma^k{}_{ij}$$

$$g^{ij} f_{;ij} = g^{ij}(f_{,ij} - f_{,k}\Gamma^k{}_{ij}) = f_{,ij}g^{ij} - f_{,k}\Gamma^k{}_{ij}g^{ij}$$

$$\vec\nabla\cdot\vec\nabla f = \operatorname{div}\operatorname{grad} f \underset{\text{cartesian coords}}{=} \delta^{ij} f_{,ij} = \left(\frac{\partial}{\partial x^1}\right)^2 + \dots + \left(\frac{\partial}{\partial x^n}\right)^2$$

$$= \nabla^2 f$$

$\boxed{\text{Laplacian}}$

$$\vec\nabla f = f^{;i}\,e_i$$

$$\nabla\vec\nabla f = \underbrace{f^{;i}{}_{;j}}\, e_i \otimes \omega^j$$

$f^{;i}{}_j$ matrix of components

$$\nabla^2 f = f^{;i}{}_i = f_{;i}{}^i = \operatorname{Tr}(f^{;i}{}_j) \quad \underline{\text{Trace of matrix}}$$

$$\operatorname{div} X = X^i{}_{;i} = |\det \underline{g}|^{-1/2}(|\det\underline{g}|^{1/2}X^i)_{,i} - C^i{}_{ik}X^k$$

$$\nabla^2 f = (g^{ik}f_{,k})_{;i} = \underbrace{|\det\underline{g}|^{-1/2}(|\det\underline{g}|^{1/2}g^{ik}f_{,k})_{,i}}_{\text{easy to evaluate}} - \underbrace{C^i{}_{ik}f^{;k}}_{=0 \text{ coord frame.}}$$

$\boxed{7.1-7.2}$ $\mathbb{R}^3$ cylindrical coords, spherical coords
6 = orthogonal curvilinear coords $\longrightarrow$ $\boxed{\text{"vector analysis"}}$

$$g = (h_1)^2\, dx^1 \otimes dx^1 + (h_2)^2\, dx^2 \otimes dx^2 + (h_3)^2\, dx^3 \otimes dx^3$$
$$= (h_1 dx^1) \otimes (h_1 dx^1) + (h_2 dx^2) \otimes (h_2 dx^2) + (h_3 dx^3) \otimes (h_3 dx^3)$$
$$= \omega^{\hat 1} \otimes \omega^{\hat 1} + \omega^{\hat 2} \otimes \omega^{\hat 2} + \omega^{\hat 3} \otimes \omega^{\hat 3}$$

$$g^{-1} = \left(\frac{1}{h_1}\frac{\partial}{\partial x^1}\right) \otimes \left(\frac{1}{h_1}\frac{\partial}{\partial x^2}\right) + \left(\frac{1}{h_2}\frac{\partial}{\partial x^2}\right) \otimes \left(\frac{1}{h_2}\frac{\partial}{\partial x^2}\right) + \left(\frac{1}{h_3}\frac{\partial}{\partial x^3}\right) \otimes \left(\frac{1}{h_3}\frac{\partial}{\partial x^3}\right)$$
$$= e_{\hat 1} \otimes e_{\hat 1} + e_{\hat 2} \otimes e_{\hat 2} + e_{\hat 3} \otimes e_{\hat 3}$$

$(\delta^{ij} e_i f) e_{\hat j}$

$$\vec\nabla f = \underbrace{\left(\frac{1}{h_1}\frac{\partial f}{\partial x^1}\right) e_{\hat 1} + \left(\frac{1}{h_2}\frac{\partial f}{\partial x^2}\right) e_{\hat 2} + \left(\frac{1}{h_3}\frac{\partial f}{\partial x^3}\right) e_{\hat 3}}_{\text{orthonormal components of gradient}} \qquad [g^{ij}\partial_j f]$$

$\omega^{\hat k}(X)$
$h_k X^k = X^{\hat k}$, $X^k = \dfrac{X^{\hat k}}{h_k}$
$X_k = (h_k)^2 X^k = h_k X^{\hat k}$

coord-expression ~~oops!~~ $h_1h_2h_3$

$$\text{Curl } X = \frac{1}{h_1h_2h_3}\, \epsilon^{ijk} \frac{\partial}{\partial x^j} (\underbrace{h_k X^{\hat k}}_{\text{no sum}}) \underbrace{e_{\hat i} h_i}_{e_i = \partial_i}$$

$$\text{div } X = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(X^{\hat 1} h_2h_3) + \frac{\partial}{\partial x^2}(X^{\hat 2} h_3h_1) + \frac{\partial}{\partial x^3}(X^{\hat 3} h_1h_2)\right]$$

$\longleftarrow |\det g|^{-1/2}(|\det g|^{1/2} X^k)_{,k}$

$h_1h_2h_3 X^{\hat 1}$

$h_2h_3 X^{\hat 1}$

$$\boxed{\begin{aligned}\det \underline{g} &= g_{11}g_{22}g_{33} \\ &= h_1^2h_2^2h_3^2 \\ |\det \underline{g}|^{1/2} &= h_1h_2h_3\end{aligned}}$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(X^{\hat 1} h_2h_3) + \frac{\partial}{\partial x^2}(X^{\hat 2} h_3h_1) + \frac{\partial}{\partial x^3}(X^{\hat 3} h_1h_2)\right]$$

ON component of grad f.

$$\nabla^2 f = \text{div grad } X = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial x_1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_3h_1}{h_2}\frac{\partial f}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial x^3}\right)\right]$$

$\boxed{7.1-7.2}$

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$$g = dx\otimes dx + dy\otimes dy + dz\otimes dz$$
$$= d\rho\otimes d\rho + \rho^2 d\phi\otimes d\phi + dz\otimes dz \qquad h_1=1,\ h_2=\rho,\ h_3=1$$
$$= dr\otimes dr + r^2 d\theta\otimes d\theta + r^2\sin^2\theta\, d\phi\otimes d\phi \qquad h_1=1,\ h_2=r,\ h_3=r\sin\theta$$

$$|\det \underline{g}|^{1/2} = h_1h_2h_3 = \rho = r^2\sin\theta$$

g^{ii}

$$\nabla^2 f = \sum_{i=1}^{3} (h_1h_2h_3)^{-1}\,\partial_i\left(h_1h_2h_3\, h_i^{-2}\,\partial_i f\right)$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\rho\,\rho^{-2}\frac{\partial f}{\partial\phi}\right)}_{\frac{1}{\rho^2}\frac{\partial^2 f}{\partial\phi^2}} + \underbrace{\frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho\frac{\partial f}{\partial z}\right)}_{\frac{\partial^2 f}{\partial z^2}} \qquad \leftarrow$$

$$= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{r^2\sin\theta}{r^2}\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{r^2\sin\theta}{r^2\sin^2\theta}\frac{\partial f}{\partial\phi}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \underbrace{\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}}_{} \qquad \leftarrow$$

$$\frac{\partial f}{\partial r} \equiv D_r f$$
$$= \cdots = \frac{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}}{r}$$
$$= \hat{r}\cdot\vec{\nabla} f \quad \text{radial derivative}$$

$$\frac{1}{r^2}\Big[\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}}_{\nabla^2 \text{ on unit sphere}}\Big]$$

$$\doteq L^2 = L_1^2 + L_2^2 + L_3^2 \qquad \text{square of angular momentum}$$

$$L_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial\phi}$$

$$\boxed{\nabla^2 = \underbrace{r^{-2}D_r(r^2 D_r)}_{\text{radial}} + \underbrace{r^{-2}L^2}_{\text{angular}}}$$

$\boxed{7.1\text{–}7.2}$

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eigenfunctions

$$L_3\, e^{im\phi} = \frac{\partial}{\partial\phi} e^{im\phi} = im\, e^{im\phi}$$

$$L_3^2\, e^{im\phi} = -m^2\, e^{im\phi}$$

m integer or not

continuous:

$$e^{im2\pi} = e^{im(0)} = 1$$

must be integer

$$\cos 2\pi m + i \sin 2\pi m$$

$$D_r\, r^n = n r^{n-1}$$

$$\underbrace{r^{-2} D_r(r^2 D_r r^n)}_{} = r^2 (n r^{n-1} r^2)_{,r}$$

$$= r^2 (n r^{n+1})_{,r} = r^{-2} n(n+1) r^n$$

$$D_r(r^2 D_r r^n) = n(n+1) r^n \quad \text{eigenfunction}$$

Laplace's eqn

$$0 = \nabla^2 \left(r^n P_{\ell m}(\theta)\, e^{im\phi} \right) \leftarrow$$

sep of variables → product of eigenfunctions

linear PDE → take linear comb of solutions

$$= \left[r^{-2} D_r(r^2 D_r) + r^{-2} L^2 \right] r^n P_{\ell m}(\theta)\, e^{im\phi}$$

$$= \left[r^{-2} n(n+1) + r^{-2} \underbrace{(-\ell(\ell+1))}_{} \right] r^n P_{\ell m}(\theta)\, e^{im\phi}$$

$$L^2 P_{\ell m}(\theta) e^{im\phi} = \left[\underbrace{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta}}_{} + \frac{1}{\sin^2\theta} \underbrace{\frac{\partial^2}{\partial\phi^2}}_{-m^2} \right] P_{\ell m}(\theta)\, e^{im\phi}$$

$$= \underbrace{\left[[-\ell(\ell+1) + m^2]\, P_{\ell m}(\theta) \right] e^{im\phi}}_{}$$

$$\lambda = -\ell(\ell+1) \qquad P_{\ell m}(\theta)\, e^{im\phi}\, e^{i\phi}$$

$$r^{-2} \left[\underbrace{n(n+1) - \ell(\ell+1)}_{} \right] r^n \underbrace{P_{\ell m}(\theta)\, e^{im\phi}}_{Y_{\ell m}(\theta,\varphi)}$$

$$= 0 \qquad \ell = n, -(n+1) \longleftarrow \underbrace{[-(n+1)]\,[\underbrace{-(n+1)+1}_{-n}]}_{= n(n+1)}$$

gen soln: $\displaystyle\sum_{n=0}^{\infty} \underbrace{[c_{n\ell m}\, r^n + d_{n\ell m}\, r^{-(n+1)}]}_{\text{radial dependence}} \underbrace{Y_{\ell m}(\theta,\varphi)}_{\text{angular dependence}}$

$\boxed{\text{spherical harmonics}}$