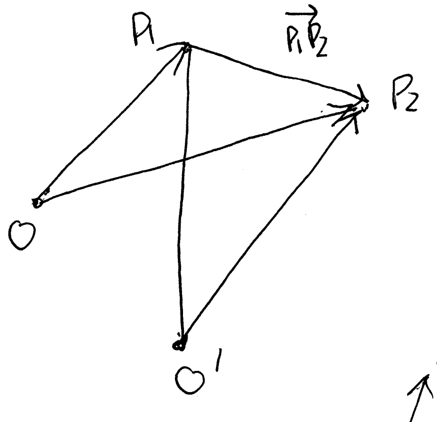
6.1–6.4

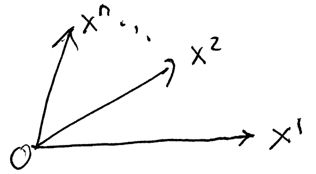
1

$\mathbb{R}^n$ as a Euclidean space is not a vector space
but a vector space modulo a choice of origin = affine space

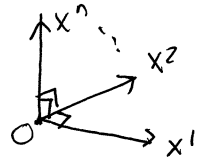


space of "difference vectors" = directed line segments
is the vector space = translation from P_1 to P_2

Pick origin O and n points $P_1 \ldots P_n$
s.t. $\overrightarrow{OP_1}, \ldots, \overrightarrow{OP_n}$ are linearly ind
get Cartesian coord system : $\overrightarrow{OP} = x^i\, \overrightarrow{OP_i}$



If you add a positive-definite inner product
you get standard orthonormal coords
on Euclidean space by choosing an
orthonormal set of
vectors $\overrightarrow{OP_i}$.



symmetry group = inhomogeneous orthogonal group

$\mathbb{R}^n$ $\quad \bar{x}^i = O^i{}_j x^j + a^i$, $\quad \underline{O}^T \underline{O} = \underline{I} \qquad O(n, \mathbb{R})$

Cartesian coordinates
(no preferred set!)

$\det \underline{O} = \pm 1 \rightarrow +1 :$ $SO(n, \mathbb{R})$ rotations
$\hookrightarrow -1$ includes reflections

translations

similar description for other signature inner products $\qquad$ "inertial coordinates"

$M^n : (- + \cdots +) \qquad \bar{x}^i = L^i{}_j x^j + a^j \qquad \underline{L}^T \underline{\eta}\, \underline{L} = \underline{\eta} = \mathrm{diag}(-1, 1, \ldots, 1)$

Poincaré transformation

Lorentz transformation = rotations, boosts, reflections

translations

all of these spaces have a $\boxed{\text{global parallelism}}$ independent of the particular inner product

in Cartesian/inertial/preferred coordinates x^i:

$\partial_i \equiv \dfrac{\partial}{\partial x^i}$ and dx^i are "constant" fields

$T = \underbrace{T^{i\ldots}{}_{j\ldots}}_{\text{constant}}\, \partial_i \otimes \cdots \otimes dx^j \otimes \cdots$ = "constant tensor field"

This gives us a standard "reference frame" in terms of which we can measure changes in nonconstant fields.

$\boxed{\text{any coordinate or noncoordinate frame:}}$

frame : $e_a = e^i{}_a \partial_i = B^i{}_a \partial_i$

dual frame : $\omega^a = \omega^a{}_i dx^i = A^a{}_i dx^i$

$\underset{\sim}{A} = \underset{\sim}{B}^{-1}$ inverse matrices

$i, j, k, \ldots$ = preferred coord indices

$a, b, c, \ldots$ = frame indices

both $= 1, \ldots, n$

$$\begin{cases} \omega^a(e_b) = \omega^a{}_i e^i{}_b = A^a{}_i B^i{}_b = \delta^a{}_b & \text{(duality)} \\ e^i{}_a \omega^a{}_j = B^i{}_a A^a{}_j = \delta^i{}_j & \text{(identity tensor components)} \end{cases}$$

$T^{a\ldots}{}_{b\ldots} = A^a{}_i \ldots A^{-1\,j}{}_b \ldots T^{i\ldots}{}_{j\ldots}$ transformation of components

$\boxed{\text{covariant derivative } \nabla}$ ≡ derivative wrt preferred coord frame

$$\underbrace{\nabla_X Y}_{\substack{\text{covariant}\\\text{derivative}\\\text{of } Y \text{ along } X}} = \nabla_{X^a e_a}(Y^b e_b) = \underbrace{X^a \nabla_{e_a}}_{\text{linearity}}(Y^b e_b) \underset{\substack{\uparrow\\\text{product}\\\text{rule}}}{=} X^a[\underbrace{(\nabla_{e_a} Y^b)}_{\substack{\text{change}\\\text{in}\\\text{components}}} e_b + Y^b \underbrace{\nabla_{e_a} e_b}_{\substack{\text{change}\\\text{in frame}}}]$$

$$\underbrace{\nabla_{e_a} e_b}_{\text{how to evaluate?}} = \Gamma^c{}_{ab} e_c \quad , \quad \Gamma^c{}_{ab} = \omega^c(\nabla_{e_a} e_b)$$ express derivatives of frame vectors in frame

$$\Gamma^c{}_{ab} e^i{}_c = \nabla_{e_a} e^i{}_b = e^j{}_a \partial_j e^i{}_b$$

just partial derivative in preferred coords. x^i of preferred components

$$\nabla_{\partial_j}(X^i \partial_i) \equiv (\partial_j X^i) \partial_i$$

constant vector field.

$$\Gamma^c{}_{ab} = \omega^c(\nabla_{e_a} e_b) = \underbrace{\omega^c{}_i}_{B^{-1c}{}_i} \underbrace{e^j{}_a \partial_j}_{e_a(} \underbrace{e^i{}_b}_{B^i{}_b)}$$

$$\underbrace{e_a(B^i{}_b)}_{dB^i{}_b(e_a)} \leftarrow \begin{matrix} Xf \\ \parallel \\ df(X) \end{matrix}$$

$$= (\underset{\sim}{B}^{-1} d\underset{\sim}{B})^c{}_b(e_a)$$

derivative index

$$\underset{\sim}{B}^{-1} d\underset{\sim}{B} \equiv \underset{\sim}{\omega} \equiv (\omega^c{}_b) = (\Gamma^c{}_{ab}\omega^a)$$ "connection" 1-form matrix

$\boxed{61-6.4}$

3

$$\nabla_X Y = X^a \left[\underbrace{(\nabla_{e_a} Y^b)}_{\substack{e_a Y^b \equiv \partial_a Y^b \\ \equiv Y^b{}_{,a}}} e_b + Y^b \underbrace{(\nabla_{e_a} e_b)}_{\Gamma^c{}_{ab} e_c} \right]$$

redundant notation

$$= X^a (Y^b{}_{,a} e_b + \Gamma^c{}_{ab} Y^b e_c)$$

switch dummy indices

$$\Gamma^b{}_{ac} Y^c e_b$$

$$= X^a (\underbrace{Y^b{}_{,a} + \Gamma^b{}_{ac} Y^c}_{Y^b{}_{;a}}) e_b = \underbrace{Y^b{}_{;a} X^a}_{[\nabla_X Y]^b} e_b$$

covariant derivative of Y

$$\equiv [\nabla Y]^b{}_a$$

$$\Rightarrow \nabla Y = [\nabla Y]^b{}_a \, e_b \otimes \omega^a = Y^b{}_{;a} \, e_b \otimes \omega^a$$

extra covariant index

$$(\nabla Y)(\cancel{\ }, X) = \nabla_X Y$$

extend first to dual frame, then all tensors using product rule

$$\delta^c{}_b = \omega^c(e_b) = \omega^c{}_i e^i{}_b = \text{product} \rightarrow \text{product rule}$$

$$0 = \nabla_{e_a} \underbrace{\delta^c{}_b}_{\text{constants}} = \nabla_{e_a} \omega^c(e_b) = (\nabla_{e_a} \omega^c)(e_b) + \underbrace{\omega^c(\nabla_{e_a} e_b)}_{\Gamma^c{}_{ab} = -(\nabla_{e_a}\omega^c)(e_b)}$$

$$\therefore \nabla_{e_a} \omega^c = -\Gamma^c{}_{ab} \omega^b$$

$$= -\omega^c{}_b(e_a)$$

$$\nabla_{e_c}: \quad T = \underbrace{T^{a\ldots}{}_{b\ldots}}_{\text{scalar}} \; \underbrace{e_a}_{\text{vector}} \otimes \cdots \otimes \underbrace{\omega^b}_{\text{covector}} \cdots$$

$$e_c T^{a\ldots}{}_{b\ldots} \qquad \Gamma^d{}_{ca} e_d \qquad -\Gamma^b{}_{cd} \omega^d$$

switch dummy indices

$$\nabla_{e_c} T = \underbrace{[T^{a\ldots}{}_{b\ldots,c} + \Gamma^a{}_{cd} T^{d\ldots}{}_{b\ldots} + \cdots - \Gamma^d{}_{cb} T^{a\ldots}{}_{d\ldots} - \cdots]}_{} e_a \otimes \cdots \otimes \omega^b \cdots \cancel{\otimes \omega^c}$$

~~extra vector argument.~~

$$\equiv T^{a\ldots}{}_{b\ldots;c} \equiv [\nabla T]^{a\ldots}{}_{b\ldots c}$$

extra covariant index at end.

representation of a matrix group $G \subset GL(n,\mathbb{R})$ on vector space V :

$\underline{A} \mapsto \rho(\underline{A}) \in GL(V)$

$\underbrace{\rho(\underline{A}\,\underline{B}) = \rho(\underline{A})\rho(\underline{B})}_{\text{becomes matrix multiplication when expressed in a basis of } V}$ "same group multiplication"

$\downarrow\ \mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{R})$

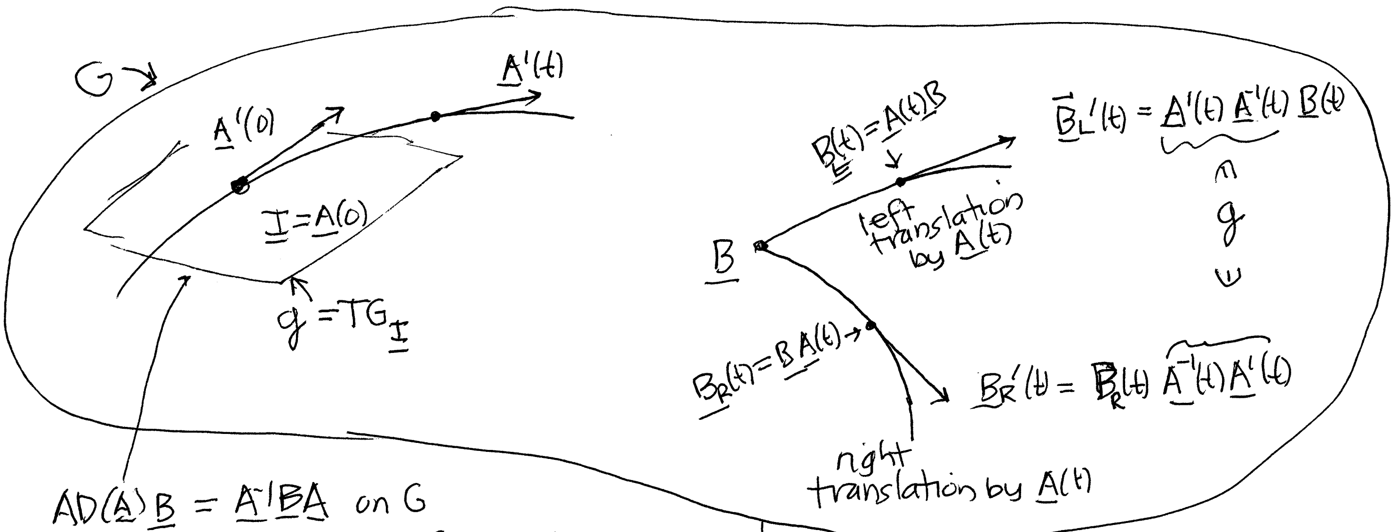
Its Lie algebra has a representation on V :

$\underline{A} \mapsto \sigma(\underline{A}) \in \mathfrak{gl}(V)$

$\sigma([\underline{A},\underline{B}]) = [\sigma(\underline{A}),\sigma(\underline{B})]$ "same commutator relations"

$\sigma(\mathfrak{g})$ = corresponding matrix Lie algebra on V when expressed in a basis of V

TEASE (no time!):



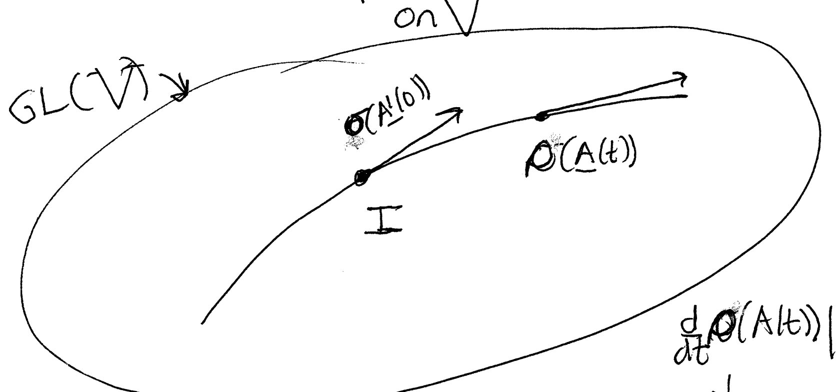
$AD(\underline{A})\underline{B} = \underline{A}^{-1}\underline{B}\underline{A}$ on G

$AD(A)\underline{I} = \underline{I}$ identity fixed point

"like generalized rotations about $\underline{I}$"

group manifold geometry

representation on V



"σ is just differential $d\rho$"

$\frac{d}{dt}\rho(A(t))\big|_{t=0} = \sigma(\underline{A}'(0)) \in \mathfrak{g}$

$= d\rho\,(\underline{A}'(0))$

differential of f

$\frac{d}{dt} f(x(t)) = df\,(x'(t))$

every G has: $\begin{cases} \text{identity representation} & \rho(\underline{A})\,\underline{X} = \underline{A}\,\underline{X} \quad \text{on } \mathbb{R}^n \\ \text{adjoint representation} & \rho(\underline{A})\,\underline{B} = \underline{A}\,\underline{B}\,\underline{A}^{-1} \quad \text{on } \mathfrak{g} \text{ its Lie algebra.} \end{cases}$

$$\equiv Ad(\underline{A})\,\underline{B}$$

$$\downarrow d$$

its Lie algebra $\mathfrak{g}$ has: adjoint representation $\sigma(\underline{A})\,\underline{B} = [\underline{A},\underline{B}] \equiv ad(\underline{A})\,\underline{B}$

ENOUGH This just sets stage for $GL(n,\mathbb{R})$ itself acting on all tensor spaces over $\mathbb{R}^n$

$$T^{a\cdots}{}_{b\cdots} \longrightarrow [\rho^{(p,q)}(\underline{A})\,T]^{a\cdots}{}_{b\cdots} = A^a{}_c \cdots A^{-1d}{}_b \cdots T^{c\cdots}{}_{d\cdots}$$

tensor transformation law

$$\bar{T}^{a\cdots}{}_{b\cdots} \qquad \underline{A}(t) \to \underline{A}'(0)$$

$$\bar{T}^{a\cdots}{}_{b\cdots}{}'(0) = [\sigma^{(p,q)}(\underline{A}'(0))\,T]^{a\cdots}{}_{b\cdots}$$

$$= A'^a{}_c(0)\,T^{c\cdots}{}_{b\cdots} + \cdots - A'^d{}_b(0)\,T^{a\cdots}{}_{d\cdots} - \cdots$$

extra terms in derivative formulas

BUT we can fully evaluate covariant derivative this way.

$\boxed{6.1-6.4}$ now back to covariant derivative for tensor field

6 $$\underbrace{T^{i\cdots}{}_{j\cdots}}_{\text{Cartesian components}} = \underbrace{B^{i}{}_{a}}_{\text{frame vector components}} \cdots B^{-1b}{}_{j} \cdots \underbrace{T^{a\cdots}{}_{b\cdots}}_{\text{frame components}} = [\rho(\underline{B})\,T]^{i\cdots}{}_{j\cdots}$$

$d \downarrow$

$[\nabla T]^{i\cdots}{}_{j\cdots;k} = [dT]^{i\cdots}{}_{j\cdots,k}$ for Cartesian frame.

$$dT^{i\cdots}{}_{j\cdots} = [\underbrace{d\rho(\underline{B})}_{\text{change in frame}}\,T]^{i\cdots}{}_{j\cdots} + [\rho(\underline{B})\,\underbrace{dT}_{\text{change in components}}]^{i\cdots}{}_{j\cdots}$$

transform back to frame components

$$[\rho(\underline{B}^{-1})\,dT]^{a\cdots}{}_{b\cdots} = [\underbrace{\rho(\underline{B}^{-1})\,d\rho(\underline{B})}\,T]^{a\cdots}{}_{b\cdots} + dT^{a\cdots}{}_{b\cdots}$$

$\hookrightarrow \sigma(\underbrace{\underline{B}^{-1}d\underline{B}}_{\in\, gl(n,\mathbb{R})}) \in gl((p,q)\text{ tensors})$

$\boxed{\rho^{-1}d\rho = \sigma!}$

$\equiv \underline{\omega}$ connection $\underline{1}$-form matrix

$\nabla T = dT$ for Cartesian components
(covariant derivative = partial derivative)

$$[\nabla T]^{a\cdots}{}_{b\cdots} = dT^{a\cdots}{}_{b\cdots} + \rho(\underline{\omega})\,T]^{a\cdots}{}_{b\cdots}$$

$$[\nabla_X T]^{a\cdots}{}_{b\cdots} = dT^{a\cdots}{}_{b\cdots}(X) + [\rho(\underline{\omega}(X))\,T]^{a\cdots}{}_{b\cdots}$$

FINAL RESULT

$$\boxed{[\nabla T]^{a\cdots}{}_{b\cdots;c} = T^{a\cdots}{}_{b\cdots,c} + \Gamma^{a}{}_{cd}T^{d\cdots}{}_{b\cdots} + \cdots - \Gamma^{d}{}_{cb}T^{a\cdots}{}_{d\cdots} - \cdots}$$

where $\omega^a{}_b = \Gamma^a{}_{cb}\omega^c = \underbrace{B^{-1a}{}_c\, dB^c{}_b}$ easy to evaluate (Maple!)

$\downarrow$
read off coefficients of ω^c in each matrix entry to get $\Gamma^a{}_{cb}$.

no metric needed!
same covariant derivative for $\mathbb{R}^n$ or M^n as long as flat space

same parallelism — same for any covariant constant metric!

$\boxed{6.1\text{-}6.4}$

Simpler story : one index

$$[Y^i = B^i{}_b Y^b] \longleftarrow \boxed{e_b = e^i{}_b \partial_i \equiv B^i{}_b \cdot \partial_i}$$

cartesian

$$\downarrow d$$

$$B^{-1a}{}_i \left[[\nabla Y]^i \equiv dY^i = B^i{}_b dY^b + dB^i{}_b Y^b \right]$$

$$[\nabla Y]^a = \underbrace{B^{-1a}{}_i B^i{}_b}_{\delta^a{}_b} dY^b + \underbrace{B^{-1a}{}_i dB^i{}_b}_{\omega^a{}_b} Y^b \qquad \underline{\omega} = \underline{B}^{-1} d\underline{B}$$

$$= dY^a + \omega^a{}_b Y^b$$

$$\underbrace{[\nabla_X Y]^a}_{Y^a{}_{;c} X^c} = \underbrace{dY^a(X)}_{\substack{X Y^a = \\ Y^a{}_{,c} X^c}} + \underbrace{\omega^a{}_b(X)}_{\Gamma^a{}_{cb} X^c} Y^b$$

$$Y^a{}_{;c} X^c = Y^a{}_{,c} X^c + \Gamma^a{}_{cb} Y^b X^c = (Y^a{}_{,c} + \Gamma^a{}_{cb} Y^b) X^c$$

$$[\nabla_X T]^{a\ldots}{}_{b\ldots} = dT^{a\ldots}{}_{b\ldots} + \underbrace{[\sigma(\underline{\omega}) T]}^{a\ldots}{}_{b\ldots}$$

just extends linear action of $\underline{\omega}$ to all the indices.

Frame invariant derivative

this is what a representation map does

Gauge invariant derivatives are similar for theories of weak & strong interactions & electromagnetism

Gauge groups: $\underline{SU(2) \times U(1)}$

electroweak : gauge bosons & photon
gauge field & EM field

$SU(3)$
stronginteractions
gauge bosons / fields

connection transformation law

$$e_a = B^i{}_a \partial_i \quad \longrightarrow \quad \omega^a{}_b = B^{-1a}{}_i \, dB^i{}_b$$

$$\underline{\omega} = \underline{B}^{-1} d\underline{B}$$

$$\underline{B} = \underline{B}_1 \underline{B}_2 :$$

$$\begin{aligned}
\underline{\omega} &= (\underline{B}_1 \underline{B}_2)^{-1} d(\underline{B}_1 \underline{B}_2) \\
&= \underline{B}_2^{-1} \underline{B}_1^{-1} \left((dB_1) B_2 + B_1 dB_2 \right) \\
&= B_2^{-1} \underbrace{(B_1^{-1} dB_1)}_{\underline{\omega}_1} B_2 + \underbrace{B_2^{-1} dB_2}_{\underline{\omega}_2}
\end{aligned}$$

$$\boxed{\underline{\omega} = \underbrace{B_2^{-1} \underline{\omega}_1 B_2}_{\substack{\text{component} \\ \text{transformation} \\ \text{like tensor}}} + \underbrace{\underline{\omega}_2}_{\substack{\text{extra} \\ \text{piece.}}}}$$

inhomogeneous transformation law for connection

one consequence is that one can make $\underline{\omega}$ zero at a point or along a curve by properly choosing $\underline{B}_2$.

(formulas simplify at such points)

$\mathbb{R}^n$, Cartesian coords $\{x^i\}$, metric $g = g_{ij}\, dx^i \otimes dx^j$ constant: $dg_{ij}=0$

(1) g is therefore covariant constant by definition

$$0=\nabla g: \quad 0 = g_{ij;k} = g_{ij,k} - g_{\ell j}\Gamma^{\ell}{}_{ki} - g_{i\ell}\Gamma^{\ell}{}_{kj}$$ true in any coord system

$$= g_{ij,k} - \Gamma_{jki} - \Gamma_{ikj}$$ index lowering

$$= g_{ij,k} - 2\Gamma_{(i|k|j)}$$ twice symmetric part, k excluded.

(2) coordinate connection components symmetric $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$

Let's rederive this fact.

$$\frac{\partial}{\partial x^{i'}} = B^j{}_{i'}\frac{\partial}{\partial x^j} = \frac{\partial x^j}{\partial x^{i'}}\frac{\partial}{\partial x^j} \quad \text{(chain rule from Cartesian coords)}$$

$$\omega^{i'}{}_{j'} = B^{-1\,i'}{}_{\ell}\,\frac{\partial}{\partial x^{k'}}(B^{\ell}{}_{j'})\,dx^{k'} \quad \leftarrow (B^{-1}dB)^{i'}{}_{j'}$$

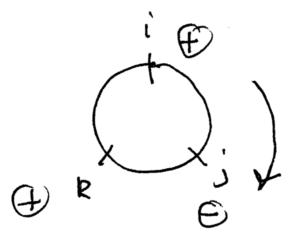
$$= \underbrace{\omega^{i'}{}_{\ell}\,\frac{\partial^2 x^{\ell}}{\partial x^{k'}\partial x^{j'}}}\,dx^{k'} = \underbrace{\Gamma^{i'}{}_{k'j'}}\,dx^{k'}$$

$$\Gamma^{i'}{}_{k'j'} = \omega^{i'}{}_{\ell}\frac{\partial^2 x^{\ell}}{\partial x^{k'}\partial x^{j'}} = \omega^{i'}{}_{\ell}\frac{\partial^2 x^{\ell}}{\partial x^{j'}\partial x^{k'}} = \Gamma^{i'}{}_{j'k'}.$$

"symmetric connection"

(3): (1)+(2) uniquely determines connection as a function of the metric!

$$g_{ij,k} = \Gamma_{jki} + \Gamma_{ikj}$$
$$-g_{jk,i} = -\Gamma_{kij} - \Gamma_{jik}$$
$$+g_{ki,j} = \Gamma_{ijk} + \Gamma_{kji}$$



anticyclic combination (extra sign)

i first · j first · k first

$$g_{ij,k} - g_{jk,i} + g_{ki,j} = (\Gamma_{ijk} + \Gamma_{ikj}) + (\underbrace{\Gamma_{jki} - \Gamma_{jik}}_{0}) + (\underbrace{\Gamma_{kji} - \Gamma_{kij}}_{0})$$ symmetry

$$= 2\Gamma_{ijk}$$

$$\Gamma_{ijk} = \tfrac{1}{2}(g_{ij,k} - g_{jk,i} + g_{ki,j})$$ Christoffel symbol first kind

$$\boxed{\Gamma^i{}_{jk} = g^{i\ell}\Gamma_{\ell jk} = \tfrac{1}{2}g^{i\ell}(g_{\ell j,k} - g_{jk,\ell} + g_{k\ell,j}) \equiv \left\{ {i \atop jk} \right\}}$$ second kind

Given any metric $g = g_{ij}\, dx^i \otimes dx^j$ in any coords in any space get connection components
$$\Gamma^i{}_{jk} = \left\{ {i \atop jk} \right\} \qquad \text{(above formula)}$$

In flatspace $\omega = \underline{B}^{-1} d\underline{B}$ is more efficient, but this works in nonflat spaces!! Can define parallel transport on nonflat spaces!

<u>Example</u> Any parametrized surface in $\mathbb{R}^3$ (or M^3)!

1) sphere of radius $r = r_0$ $(dr = 0)$ = coordinate surface in spherical coords: $S(r_0)$

<u>metric</u>
$$ds^2 = dx^2 + dy^2 + dz^2 = g_{rr}\, dr^2 + 2(g_{r\theta}\, d\theta + g_{r\varphi}\, d\varphi)\, dr$$
$$+ \left\{ g_{\theta\theta}\, d\theta^2 + 2 g_{\theta\varphi}\, d\theta\, d\varphi + g_{\varphi\varphi}\, d\varphi^2 \right\}$$
$$= ds^2_{S(r_0)} = ds^2\big|_{\substack{r=r_0 \\ dr=0}} \quad \text{"ignore } r\text{"}$$

<u>connection</u>: Let $A, B, C = \theta, \varphi$ $\longrightarrow$ $= g_{AB}\, dx^A\, dx^B$

3 r's $\quad \Gamma^r{}_{rr}$.

2 r's $\quad \Gamma^r{}_{Ar},\ \Gamma^r{}_{rA},\ \Gamma^A{}_{rr}$

1 r $\quad \Gamma^r{}_{AB},\ \Gamma^A{}_{rB}$

0 r $\quad \boxed{\Gamma^A{}_{BC}} \longleftarrow$ $\boxed{\text{formula with no } r \text{ index} = \text{connection of metric on sphere}}$

2) start from scratch like any surface (parametrized!)

$x^i = x^i(u^1, u^2)$

$\dfrac{\partial x^i}{\partial u^A} \equiv E^i{}_A\,, \quad dx^i = E^i{}_A\, du^A \qquad A, B = 1,2$ parameter/coord indices on sphere/whatever

$$ds^2 = \delta_{ij}\, dx^i dx^j = \delta_{ij} (E^i{}_A\, du^A)(E^j{}_B\, du^B)$$
$$= \underbrace{(\delta_{ij}\, E^i{}_A E^j{}_B)}_{g_{AB} = g(E_A, E_B) = \vec{E}_A \cdot \vec{E}_B}\, du^A\, du^B \qquad \text{(dot products of Cartesian component vectors)}$$

sphere $= r_0^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{ds^2_{S(1)} \text{ unit sphere.}}$

$\boxed{6.5\text{-}6.6}$

3

$$\Gamma^{C}{}_{AB} = \tfrac{1}{2} g^{CC}(-g_{AB,C} + g_{AC,B} + g_{BC,A}) \Leftarrow \begin{matrix} g_{\theta\theta} = r_0^2 \\ g_{\varphi\varphi} = r_0^2 \sin^2\theta \end{matrix}$$

$$\Gamma^{\theta}{}_{\theta\varphi} = \tfrac{1}{2} g^{\theta\theta}(-\underbrace{\cancel{g_{\theta\varphi,\theta}}}_{0} + \underbrace{\cancel{g_{\theta\theta,\varphi}}}_{0} + \underbrace{\cancel{g_{\varphi\theta,\theta}}}_{0}) = 0$$

$$\Gamma^{\varphi}{}_{\varphi\theta} = \tfrac{1}{2} g^{\varphi\varphi}(-\underbrace{\cancel{g_{\varphi\theta,\varphi}}}_{0} + g_{\varphi\varphi,\theta} + \underbrace{\cancel{g_{\theta\varphi,\varphi}}}_{0}) = \tfrac{1}{2}(\ln g_{\varphi\varphi})_{,\varphi} = (\ln(g_{\varphi\varphi}{}^{1/2}))_{,\theta}$$

$$= \frac{\cos\theta}{\sin\theta} = \boxed{\cot\theta}$$

$$\Gamma^{\varphi}{}_{\theta\theta} = \tfrac{1}{2} g^{\varphi\varphi}(-\underbrace{\cancel{g_{\theta\theta,\varphi}}}_{0} + \underbrace{\cancel{g_{\theta\varphi,\theta}}}_{0} + \underbrace{\cancel{g_{\theta\varphi,\theta}}}_{0}) = 0$$

$$\Gamma^{\theta}{}_{\varphi\varphi} = \tfrac{1}{2} g^{\theta\theta}(-g_{\varphi\varphi,\theta} + \underbrace{\cancel{g_{\varphi\theta,\varphi}}}_{0} + \underbrace{\cancel{g_{\varphi\theta,\varphi}}}_{0}) = \frac{1}{2r_0^2}(-r_0^2\, 2\sin\theta\cos\theta)$$

$$= \boxed{-\sin\theta\cos\theta.}$$

$$(\Gamma^{\theta}{}_{\theta\theta} = 0 = \Gamma^{\varphi}{}_{\varphi\varphi})$$

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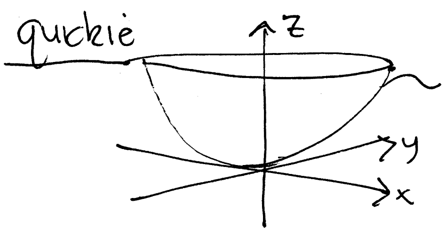
other examples: 1) surfaces of revolution (like sphere)
independent of φ, function only of 1 coordinate.
circular orbits on ~~surface~~ surface.
2) screw rotation symmetry — helical orbits on surface

<u>explore in chapter 8</u>

quickie



$z = \rho^2$ in cyl coords (ρ, φ, z)

$$\vec{r} = \langle \rho\cos\varphi, \rho\sin\varphi, \rho^2 \rangle$$

$$\vec{E}_\rho = \frac{\partial \vec{r}}{\partial \rho} = \langle \cos\varphi, \sin\varphi, 2\rho \rangle$$

$$\vec{E}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} = \langle -\rho\sin\varphi, \rho\cos\varphi, 0 \rangle$$

$$\left.\begin{aligned} \vec{E}_\rho\cdot\vec{E}_\rho &= \cos^2\varphi + \sin^2\varphi + 4\rho^2 = 1 + 4\rho^2 \\ \vec{E}_\rho\cdot\vec{E}_\varphi &= 0 \\ \vec{E}_\varphi\cdot\vec{E}_\varphi &= \rho^2(\sin^2\varphi + \cos^2\varphi) = \rho^2 \end{aligned}\right\} \quad ds^2 = \underbrace{(1+4\rho^2)}_{g_{\rho\rho}} d\rho^2 + \underbrace{\rho^2}_{g_{\varphi\varphi}} d\varphi^2$$

orthogonal coor system.

$\boxed{6.5\text{-}6.6}$

4

But orthonormal frames most useful for interpretation.

$$g = g_{ab}\,\omega^a \otimes \omega^b \qquad (g_{ab}) = \mathrm{diag}(\pm 1, \ldots, \pm 1)\,, \quad dg_{ab} = 0$$

metric components are constant.

derivatives are packaged in frame derivatives

Lie bracket: $[X,Y]^k = XY^k - YX^k = Y^k{}_{,i}X^i - X^k{}_{,i}Y^i$ comma to semi-colon

corresponding covariant derivative

$$[\nabla_X Y - \nabla_Y X]^k = Y^k{}_{;i}X^i - X^k{}_{;i}Y^i$$

$$= (Y^k{}_{,i} + \Gamma^k{}_{ij}Y^j)X^i - (X^k{}_{,i} + \Gamma^k{}_{ij}X^j)Y^i$$

$$= \underbrace{Y^k{}_{,i}X^i - X^k{}_{,i}Y^i}_{[X,Y]^k} + \underbrace{\Gamma^k{}_{ij}X^iY^j - \underbrace{\Gamma^k{}_{ij}X^jY^i}_{\Gamma^k{}_{ji}X^iY^j}}_{\underbrace{(\Gamma^k{}_{ij} - \Gamma^k{}_{ji})}_{=0}X^iY^j}$$

$=0$ symmetry condition

$$\boxed{\nabla_X Y - \nabla_Y X = [X,Y]}$$

coordinate independent.

In a frame: $[e_a, e_b] = C^c{}_{ab}e_c$ "structure functions" of frame

$$\nabla_{e_a} e_b - \nabla_{e_b} e_a = [e_a, e_b]$$

$$\Gamma^c{}_{ab}e_c - \Gamma^c{}_{ba}e_c = C^c{}_{ab}e_c \qquad \Gamma^c{}_{ab} - \Gamma^c{}_{ba} = C^c{}_{ab}$$

$$\boxed{2\Gamma^c{}_{[ab]} = C^c{}_{ab}}$$

now antisymmetric part tied to structure functions

lower index $\boxed{\Gamma_{cab} - \Gamma_{cba} = C_{cab}}$

$$\Gamma_{cab} - C_{cab} = \Gamma_{cba}$$ for next page.

$\boxed{6.5\text{–}6.6}$

5

Repeat $0 = \nabla g$:

$$0 = g_{ab;c} = g_{ab,c} - g_{da}\Gamma^{d}{}_{cb} - g_{db}\Gamma^{d}{}_{ca}$$

$$= g_{ab,c} - \Gamma_{acb} - \Gamma_{bca}$$

$$g_{ab,c} = \Gamma_{acb} + \Gamma_{bca}$$

$$\boxed{\begin{array}{l}\text{if } g_{ab,c} = 0 \\ \Gamma_{(a|c|b)} = 0\end{array}}$$

$$\boxed{\begin{array}{ll}\text{connection} & \omega_{ab} = \Gamma_{acb}\,\omega^{c} \\ \text{1-form:} & \omega_{(ab)} = \Gamma_{(a|c|b)}\,\omega^{c} = 0 \\ & \text{antisymmetric}\end{array}}$$

$$g_{ab,c} = \Gamma_{acb} + \Gamma_{bca}$$

$$-g_{bc,a} = -\Gamma_{cab} - \Gamma_{bac}$$

$$+g_{ca,b} = \Gamma_{cba} + \Gamma_{abc}$$

a first   b first   c first

$$\underbrace{g_{ab,c} - g_{bc,a} + g_{ca,b}}_{\text{anticyclic}} = (\Gamma_{acb} + \Gamma_{abc}) + \underbrace{(\Gamma_{bca} - \Gamma_{bac})}_{= C_{bca}} - \underbrace{(\Gamma_{cab} - \Gamma_{cba})}_{= C_{cab}}$$

$$= \Gamma_{abc} - C_{abc}$$

$$= 2\Gamma_{abc} - (\underbrace{C_{abc} - C_{bca} + C_{cab}}_{\text{anticyclic}})$$

solve

$$\Gamma_{abc} = \tfrac{1}{2}(g_{ab,c} - g_{bc,a} + g_{ca,b}) + \tfrac{1}{2}(C_{abc} - C_{bca} + C_{cab})$$

$$\Gamma^{a}{}_{bc} = \tfrac{1}{2}g^{ad}(g_{db,c} - g_{bc,d} + g_{cd,b}) + \tfrac{1}{2}(C^{a}{}_{bc} - \underbrace{C_{bc}{}^{a}}_{+C_{b}{}^{a}{}_{c}} + C_{c}{}^{a}{}_{b})$$

$$\boxed{\Gamma^{a}{}_{bc} = \underbrace{\left\{ {a \atop bc} \right\}}_{\to 0} + \tfrac{1}{2}C^{a}{}_{bc} + C_{(b}{}^{a}{}_{c)}}$$

general formula in any frame.

orthonormal frame

$$\boxed{\Gamma^{a}{}_{bc} = \tfrac{1}{2}C^{a}{}_{bc} + C_{(b}{}^{a}{}_{c)}}$$

structure functions contain geometry.

$\boxed{6.5-6.6}$

6

In orthonormal coordinates – easy to evaluate Lie brackets : $e_i = (g_{ii})^{-1/2}\partial_i$

<u>example</u>: spherical coordinates $\quad g_{rr}=1,\ g_{\theta\theta}=r^2,\ g_{\varphi\varphi}=r^2\sin^2\theta$

$$e_r=\partial_r,\quad e_\theta = r^{-1}\partial_\theta,\quad e_\varphi = r^{-1}(\sin\theta)^{-1}\partial_\varphi$$

$$[e_r, e_\theta] = [\partial_r, r^{-1}\partial_\theta] = -r^{-2}\partial_\theta = -\frac{1}{r}e_\theta$$

$$[e_r, e_\varphi] = [\partial_r, r^{-1}(\sin\theta)^{-1}\partial_\varphi] = -r^{-2}(\sin\theta)^{-1}\partial_\varphi = -\frac{1}{r}e_\varphi$$

$$[e_\theta, e_\varphi] = [r^{-1}\partial_\theta, r^{-1}(\sin\theta)^{-1}\partial_\varphi] = -r^{-2}(\sin\theta)^{-2}\cos\theta\,\partial_\varphi = -\frac{\cot\theta}{r}e_\varphi$$

$$C^\theta{}_{r\theta} = -r^{-1} = C^\varphi{}_{r\varphi},\quad C^\varphi{}_{\theta\varphi} = -r^{-1}\cot\theta$$

$$\underline{\omega} = \begin{bmatrix} 0 & \omega_{r\theta} & \omega_{r\varphi} \\ -\omega_{r\theta} & 0 & \omega_{\theta\varphi} \\ -\omega_{r\varphi} & \omega_{\theta\varphi} & 0 \end{bmatrix}$$

antisymmetric

$$\Gamma_{r\theta\theta}(r\,d\theta) = \frac{1}{2}(\underbrace{C_{r\theta\theta}}_{0} + 2\underbrace{C_{\theta r\theta}}_{-r^{-1}})\,r\,d\theta = -d\theta$$

$$\Gamma_{r\varphi\varphi}(r\sin\theta\,d\varphi) = \frac{1}{2}(\underbrace{C_{r\varphi\varphi}}_{0} + 2\underbrace{C_{\varphi r\varphi}}_{-r^{-1}})\,r\sin\theta\,d\varphi = -\sin\theta\,d\varphi$$

$$\Gamma_{\theta\varphi\varphi}(r\sin\theta\,d\varphi) = \frac{1}{2}(C_{\theta\varphi\varphi} + 2\underbrace{C_{\varphi\theta\varphi}}_{-r^{-1}\cot\theta})\,r\sin\theta\,d\varphi$$

$$= -\cos\varphi\,d\varphi$$

$$= \begin{bmatrix} 0 & -d\theta & -\sin\theta\,d\varphi \\ d\theta & 0 & -\cos\theta\,d\varphi \\ \sin\theta\,d\varphi & \cos\theta\,d\varphi & 0 \end{bmatrix}$$

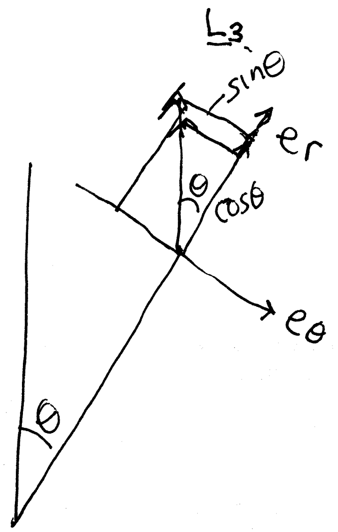
$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d\theta + (\underline{L}_1\cos\theta\,\cancel{d\varphi} - \underline{L}_2\sin\theta)\,d\varphi \equiv \Omega^a \underline{L}_a$$

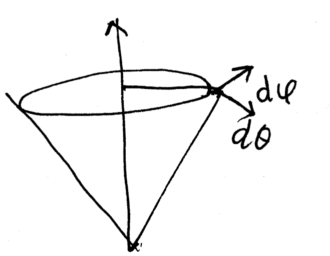
$$\vec{\Omega} = \langle\cos\theta, -\sin\theta, 0\rangle\,d\varphi + \langle 0,0,1\rangle\,d\theta$$

$$\Omega = (\underbrace{\cos\theta\,e_r - \sin\theta\,e_\theta}_{e_z})\,d\varphi + e_\varphi\,d\theta$$

incremental rotation about vertical

incremental rotation in e_r–e_θ plane





$\boxed{6.5\text{-}6.6}$

7

<u>example</u> dx is covariant constant (in spherical ON frame).

$$\begin{cases} \omega^r = dr \\ \omega^\theta = r\,d\theta \\ \omega^\varphi = r\sin\theta\, d\theta \end{cases}$$

$$x = r\sin\theta\cos\varphi$$

$$X^k = dx = dr\sin\theta\cos\varphi + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi$$

$$= \sin\theta\cos\varphi\ \omega^r + \cos\theta\cos\varphi\ \omega^\theta - \sin\varphi\ \omega^\varphi$$

$$\hookrightarrow \langle \sin\theta\cos\varphi,\ \cos\theta\cos\varphi,\ -\sin\varphi \rangle = \langle X_i \rangle$$

$$\nabla X_i = dX_i - X_j\,\omega^j{}_i$$

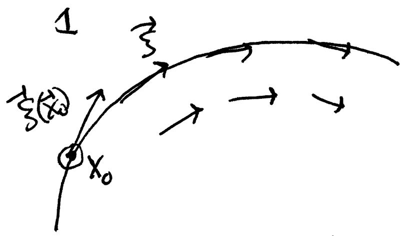
$$= d\langle \sin\theta\cos\varphi,\ \cos\theta\cos\varphi,\ -\sin\varphi \rangle - \langle \sin\theta\cos\varphi,\ \cos\theta\cos\varphi,\ -\sin\varphi \rangle \begin{bmatrix} 0 & -d\theta & -\sin\theta\, d\varphi \\ d\theta & 0 & -\cos\theta\, d\varphi \\ \sin\theta\, d\varphi & \cos\theta\, d\varphi & 0 \end{bmatrix}$$

$$= \langle \cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi,\ -\sin\theta\cos\varphi\, d\theta - \cos\theta\sin\varphi\, d\varphi,\ \cos\varphi\, d\varphi \rangle$$

$$- \langle \cos\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi,\ -\sin\theta\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\varphi,\ -\sin^2\theta\cos\varphi\, d\varphi - \cos^2\theta\cos\varphi\, d\varphi + 0 \rangle$$

$$= 0 \ \checkmark$$

$\boxed{6.7\text{–}6.8}$ Lie bracket: what is it? $\boxed{\text{Section } 5.3}$



$$\frac{dx^i}{dt} = \xi^i(x) \longrightarrow x^i = f^i_\xi(x_0,t)$$

$$f^i_\xi(x,t) = e^{t\xi(x)}\, x^i$$
$$= \left[1 + t\xi(x) + \frac{t^2}{2}\xi(x)^2 + \dots\right] x^i$$
$$= x^i + t\xi^i(x) + \frac{t^2}{2}\xi^2(x)\, x^i + \dots$$
$$= x^i(f_\xi(x,t))$$

$$x^i \longrightarrow \bar{x}^i = f^i_\xi(x,t)$$

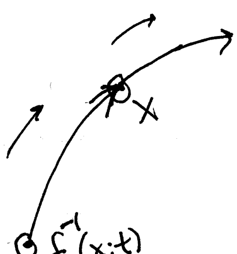
1-parameter group of transformations (Abelian parameter t)

generalizes to any functions $x^i \longrightarrow F$

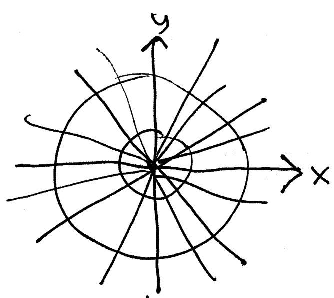
$$\underbrace{F(\underbrace{f_\xi(x,t)}_{\text{new pt}})}_{\text{value of } F \text{ at new point}} = e^{t\xi(x)} F(x)$$

dragged along field:

$$\underset{\uparrow\ \text{new pt}}{\bar{F}_t(x)} = F(\underbrace{f_\xi(x,-t)}_{\text{old point sent to } x \,=\, \text{inverse transformation}}) = e^{-t\xi(x)} F(x)$$

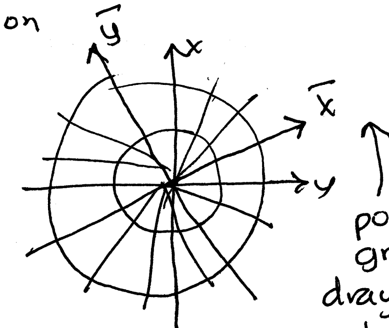


$f^{-1}_\xi(x;t)$
$= f_\xi(x;-t)$



example

rotate by 30° including polar grid

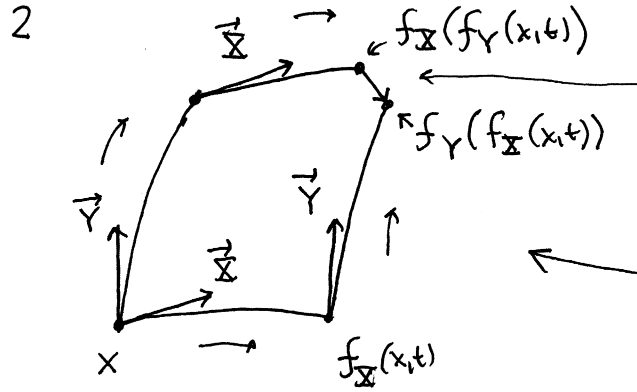


polar grid dragged along.

values of functions dragged along

$\boxed{6.7\text{-}6.8}$

2



failure of curvilinear parallelogram to close

move along flowlines of X & Y in opposite orders

DIFFERENCE

$$f_Y^i(f_X(x,t),t) - f_X^i(f_Y(x,t),t)$$

$$= e^{tX}\underbrace{f_Y^i(x,t)}_{e^{tY}x^i} - e^{tY}\underbrace{f_X^i(x,t)}_{e^{tX}x^i} = (e^{tX}e^{tY} - e^{tY}e^{tX})\,x^i$$

$$= (1 + tX + \tfrac{t^2}{2}X^2 + \dots)(1 + tY + \tfrac{t^2}{2}Y^2 + \dots)\,x^i$$
$$- (1 + tY + \tfrac{t^2}{2}Y^2 + \dots)(1 + tX + \tfrac{t^2}{2}X^2 + \dots)\,x^i$$

expand, keep only terms up to order t^2

$$\begin{aligned}
= {}&① \quad (1 + tY + \tfrac{t^2}{2}Y^2 + \dots)x^i - (1 + tX + \tfrac{t^2}{2}X^2 + \dots)x^i \\
&② + (tX + t^2XY + \dots)x^i - (tY + t^2 YX \dots)x^i \\
&③ + (\tfrac{t^2}{2}X^2 + \dots)x^i - (\tfrac{t^2}{2}Y^2 + \dots)x^i
\end{aligned}$$

$$= t^2(XY - YX)\,x^i + \dots = t^2[X,Y]\,x^i + \dots$$

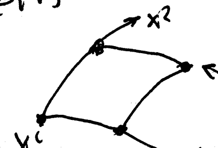
$$= t^2[X,Y]^i + \dots$$

$\downarrow$

$t \to 0$; difference goes to 0 like t^2 compared to sides of "parallelogram" which are linear in t

Lie bracket measures failure to close at second order

If $[X,Y] = 0 \rightarrow$ can find coords s.t. $X = \partial/\partial x^1$, $Y = \partial/\partial x^2$



parallelogram closes to all orders in t — coordinate grid box!

$(x^1, x^2) \rightarrow (x^1 + a, x^2 + b)$

$[X,Y]\neq 0$ means 1-parameter groups dont commute

FACT: $\{E_a\}$ r-dim Lie algebra of vector fields on $\mathbb{R}^n$ closed under Lie bracket

$$[E_a, E_b] = \underset{\uparrow\ \text{constants}}{C^c_{ab}}\, E_c$$

← (must be constant linear combination to be closed as r-dim subspace of ∞-dim space of vector fields)

"structure constant tensor" on Lie algebra

THEN $\quad x^i \to e^{\theta^a E_a} x^i = f^i(x,\theta)$ = group of transformations of $\mathbb{R}^n$ into itself

"group parameters"

$x^i \to e^{t\theta^a E_a} x^i$ are all 1-d Abelian subgroups but Lie brackets determine noncommutivity of distinct such subgroups.

<u>EXAMPLE</u>
$\mathbb{R}^3$

$L_i = \epsilon_{ijk} x^j \partial_k$ generate rotations about origin

$p_j = \partial_j$ generate translations (Abelian group)

$[L_i, L_j] = -\epsilon_{ijk} L_k$,

$[p_i, p_j] = 0$ ← abelian subalgebra → generates

$[L_i, p_j] = -\epsilon_{ijk} p_k$

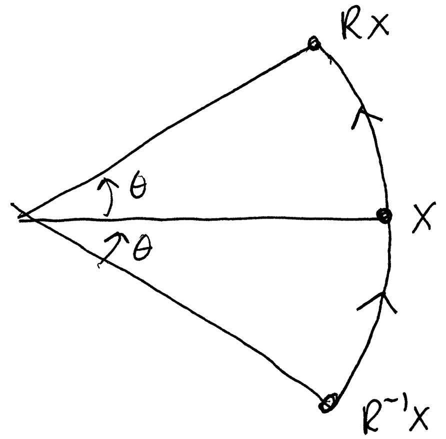
$x^i \to R^i{}_j x^j + b^i$ Euclidean group of symmetries of Euclidean space

$e^{t\theta^a L_a} x^i = [(e^{t\theta^a \underline{L}_a})\underline{x}]^i$ corresponding rotation matrix

$e^{b^j\partial_j} x^i = [1 + b^j\partial_j + \underbrace{b^j\partial_j b^k\partial_k}_{0} + \ldots]\, x^i = x^i + b^i + \underline{\underline{0}}$

How to drag along vector fields by a transformation?

Think rotation R



functions are easy:

$F(x)$ = function = scalar field

rotate function

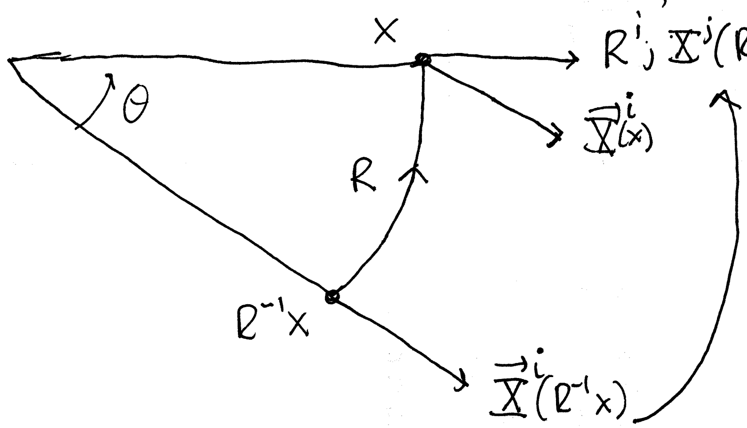
$$(RF)(x) \equiv F(R^{-1}x)$$

rotated field at x — value of old field at $R^{-1}x$

but with vector fields we also have to rotate their components, as well as drag along their values from $R^{-1}x$

$$x^i \to \bar{x}^i = R^i{}_j x^j$$

$$\frac{\partial \bar{x}^i}{\partial x^j} = R^i{}_j$$



actively rotate components evaluated at inverse transformation pt

$$R^i{}_j \, X^j(R^{-1}x) \equiv \underbrace{(RX)}_{\text{new field at } x}{}^i(x)$$

$\vec{X}^i(x)$

$\vec{X}^i(R^{-1}x)$

" dragged along field "

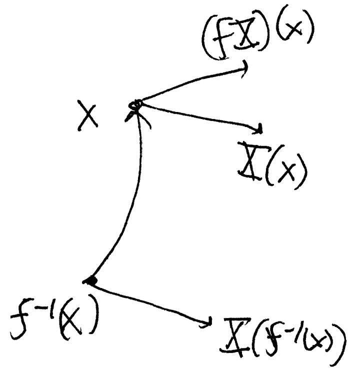
$\boxed{6.7-6.8}$

5

In general $x^i \to \bar{x}^i = f^i(x)$

$$X^i(x) \longrightarrow \underbrace{\frac{\partial \bar{x}^i}{\partial x^j}}_{\substack{\text{Jacobian} \\ \text{matrix} \\ \text{transforms} \\ \text{components}}}(f^{-1}(x))\, X^j(f^{-1}(x)) \equiv (fX)^i(x)$$

values come from pt sent to x by f



like a coord transformation but "active"
defines a new vector field

extend to all tensors:

$$T^{i\dots}{}_{j\dots}(x) \to (fT)^{i\dots}{}_{j\dots} = \frac{\partial \bar{x}^i}{\partial x^m}\cdots\frac{\partial x^n}{\partial \bar{x}^j}\cdots T^{m\dots}{}_{n\dots}(f^{-1}(x))$$

also

---

now let $f^i(x) = e^{t\xi} x^i$ be 1-parameter group of vector field ξ
"flowing along flow lines of ξ"

"xi"

Define $$[\mathcal{L}_\xi T]^{i\dots}{}_{j\dots} = -\frac{d}{dt}\Big|_{t=0} [\underbrace{e^{t\xi}}_{\text{dragged along field.}} T]^{i\dots}{}_{j\dots}$$

$$\bar{x}^i = f^i(x,t) = e^{t\xi} x^i = (1 + t\xi + \tfrac{t^2}{2}\xi^2 + \dots)\, x^i = x^i + t\xi^i + \tfrac{t^2}{2}(\xi^2)x^i + \dots$$

$$\frac{\partial \bar{x}^i}{\partial x^j} = \dots\dots = \delta^i_j + t\xi^i{}_{,j} + \tfrac{t^2}{2}\dots$$

3 kinds of terms

$$\frac{d}{dt}\Big|_{t=0}\frac{\partial \bar{x}^i}{\partial x^j} = 0 + \xi^i{}_{,j} + 0\dots = \xi^i{}_{,j}$$

$$\frac{d}{dt}\Big|_{t=0}\frac{\partial x^i}{\partial \bar{x}^j} = \dots = -\xi^i{}_{,j}$$

inverse matrix has opp sign derivative.

$$\frac{d}{dt}\Big|_{t=0} T^{i\dots}{}_{j\dots}(e^{-t\xi}x) = \dots = -\xi T^{i\dots}{}_{j\dots}$$

$\boxed{6.7-6.8}$

6

$$[\mathcal{L}_\xi T]^{i\ldots}{}_{j\ldots} = -\frac{d}{dt}[e^{t\xi}T]^{i\ldots}{}_{j\ldots}$$

$$= +\underbrace{\xi T^{i\ldots}{}_{j\ldots}}_{T^{i\ldots}{}_{j\ldots,k}\xi^k} - \xi^i{}_{,k}T^{k\ldots}{}_{j\ldots} - \ldots$$

$$+ \xi^k{}_{,j}T^{i\ldots}{}_{k\ldots} + \ldots$$

$$= \underbrace{\xi T^{i\ldots}{}_{j\ldots}}_{\text{"ordinary" derivative" along vectorfield } \xi} - \boxed{\underbrace{\sigma}_{\text{representation of } gl(n,\mathbb{R})}}(\xi^i{}_{,k})T]^{i\ldots}{}_{j\ldots}$$

opposite sign to covariant derivative terms

lowest tensors:

$\mathcal{L}_\xi f = \xi f$ functions

$\mathcal{L}_\xi X^i = \xi X^i - \xi^i{}_{,k}X^k = X^i{}_{,k}\xi^k - \xi^i{}_{,k}X^k$

$= [\xi, X]^i$ vectorfield Lie bracket.

$\mathcal{L}_\xi X_i = \xi X_i + \xi^k{}_{,i}X_k = X_{i,k}\xi^k + \xi^k{}_{,i}X_k$ 1-form

$\mathcal{L}_\xi g_{ij} = \underbrace{\xi g_{ij}}_{g_{ij,k}\xi^k} + g_{kj}\xi^k{}_{,i} + g_{ik}\xi^k{}_{,j}$ metric Lie derivative.

---

when a field is <u>invariant</u> under dragging along by $e^{t\xi}$

$e^{t\xi}T = T$ then its Lie derivative is zero

$\boxed{\mathcal{L}_\xi T = 0}$ symmetry of T

6.7–6.8

BUT "comma to semi-colon rule" holds for symmetric connection

$$\mathcal{L}_\xi X^i = X^i{}_{,k}\xi^k - \xi^i{}_{,k}X^k$$

$$= (X^i{}_{;k} - \Gamma^i{}_{kj}X^j)\xi^k - (\xi^i{}_{;k} - \Gamma^i{}_{kj}\xi^j)X^k$$

$$= X^i{}_{;k}\xi^k - \xi^i{}_{;k}X^k - \cancelto{0}{\underbrace{\Gamma^i{}_{kj}}_{\substack{=\Gamma^i{}_{jk}\\ \text{symmetric}}}\underbrace{(X^j\xi^k - X^k\xi^j)}_{\text{antisym } jk}}$$

$$= X^i{}_{;k}\xi^k - \xi^i{}_{;k}X^k$$

same formula with covariant derivative.

apply to metric g_{ij} where $g_{ij;k}=0$ covariant constant

$$(\mathcal{L}_\xi g)_{ij} = g_{ij,k} + \xi^k{}_{,i}\,g_{kj} + \xi^k{}_{,j}\,g_{ik}$$

$$= \underbrace{g_{ij;k}}_{=0} + \underbrace{\xi^k{}_{;i}\,g_{kj}}_{=(g_{jk}\xi^k)_{;i}} + \underbrace{\xi^k{}_{;j}\,g_{ik}}_{(g_{ik}\xi^k)_{;j}} \qquad \text{(can check)}$$

If invariant

$$= \xi_{j;i} + \xi_{i;j} = 2\,\xi_{(i;j)} = 0$$

twice symmetric part

$\to \nabla\xi$ is antisymmetric (when index lowered)

FLAT SPACE

→ orthogonal transformations $x^i \to O^i{}_j x^j$ Cartesian coords

matrix Lie algebra $\underline{K} = (K^i{}_j)$: $K_{ij} + K_{ji} = 0$

vector field generators $\xi = K^i{}_j x^j \partial_i$

$\xi^i{}_{,j} = K^i{}_j$ constant

symmetries of metric

$\xi_{i,j} = K_{ij}$

$\xi_{i,j} + \xi_{j,i} = K_{ij} + K_{ji} = 0$

→ translations: $\xi = \partial_i$

$\xi^i{}_{,j} = 0$ constant vector field → zero derivative

→ generating vector fields called $\boxed{\text{Killing vector fields}}$