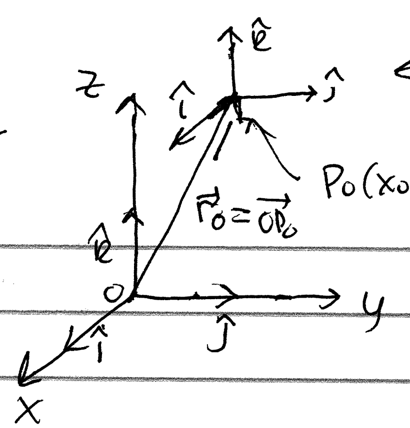
multivariable calculus :

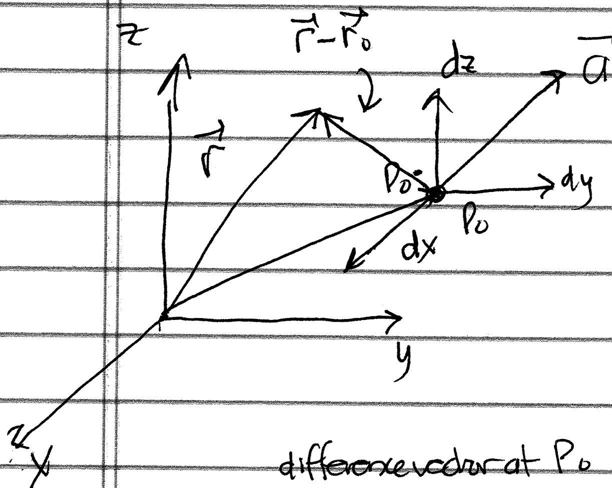
$\boxed{5.1\text{-}2}$ 1 $\mathbb{R}^3$ tangent spaces



← local $\hat{i}\hat{j}\hat{k}$ axes. at each point.

tangent space ~~at~~ P_0
= set of all difference vectors, $\vec{r}_0$ fixed

tangent vectors
= difference vectors



$\vec{r}-\vec{r}_0 = \langle x-x_0, y-y_0, z-z_0 \rangle$

$\equiv \underbrace{\langle dx|_{P_0}, dy|_{P_0}, dz|_{P_0} \rangle}$

multivariable calc "definition" of differential coordinates

new axes at each point P_0 measuring difference Cartesian coordinates.

pick out difference components of vectors wrt $\vec{r}_0$.

$\underbrace{\text{difference vector at } P_0}$

$dx|_{P_0}(\langle a^1, a^2, a^3 \rangle) = a^1$

$dy|_{P_0}(\langle a^1, a^2, a^3 \rangle) = a^2$

$dz|_{P_0}(\langle a^1, a^2, a^3 \rangle) = a^3$

$\vec{a} = \langle a^1, a^2, a^3 \rangle = a^1 \vec{e}_1|_{P_0} + a^2 \vec{e}_2|_{P_0} + a^3 \vec{e}_3|_{P_0}$

i, j, k with initial point at P_0

basis of tangent space at P_0

dual basis : $(x^1, x^2, x^3) \equiv (x, y, z)$

$dx^i_{P_0}(\vec{e}_j|_{P_0}) = \delta^i_j$ on each tangent space

$\updownarrow \qquad \updownarrow$

$x^i(e_j) = \delta^i_j$ · on whole vector space differential of a function:

$$df|_{P_0} = f_x(x_0 y_0 z_0)\, dx|_{P_0} + f_y(x_0 y_0 z_0)\, dy|_{P_0} + f_z(x_0 y_0 z_0)\, dz|_{P_0}$$
$$= f_i(P_0)\, dx^i|_{P_0}$$

tangent vector at P_0 : $\vec{X} = X^i \vec{e}_i|_{P_0}$

$$\underbrace{df|_{P_0}(\vec{X})}_{\substack{\text{linear}\\ \text{function of}\\ \text{tangent vector}\\ = \text{covector}}} = f_i(P_0)\, \underbrace{dx^i|_{P_0}(\vec{X})}_{X^i} = X^i \underbrace{\partial_i f}_{\frac{\partial f}{\partial x^i} = f_{,i} = f_i \text{ notation}}(P_0)$$

$$df|_{P_0}(\vec{e}_i|_{P_0}) = f_j(P_0)\, \underbrace{dx^j|_{P_0}(\vec{e}_i|_{P_0})}_{\delta^j{}_i} = \underbrace{f_i}_{\substack{\text{components}\\ \text{of covector.}}}(P_0)$$

$\|$

$$X^i \frac{\partial}{\partial x^i} f \Big|_{P_0}$$

$$X^i \vec{\nabla}_i f$$

$$\vec{X}\cdot\underbrace{\vec{\nabla} f}_{\text{gradient}} = \delta_{ij} X^i \underbrace{(\vec{\nabla} f)^j}_{\text{vector}} = X^i \underbrace{(\vec{\nabla} f)_i}_{\text{covector}}$$

just evaluation of covector on vector

false geometry

directional derivative $\quad \nabla_{\hat{u}} f = \hat{u}\cdot\vec{\nabla} f$ = derivative of f along direction $\hat{u}$

unit vector = ~~geometry~~

can't rely on dot product when just linearity needed

extend to any vector

$$\nabla_{\vec{X}} f = X^i \frac{\partial f}{\partial x^i} = \text{derivative along tangent vector}$$

$$\equiv \left(X^i \frac{\partial}{\partial x^i}\right) f \equiv X f$$

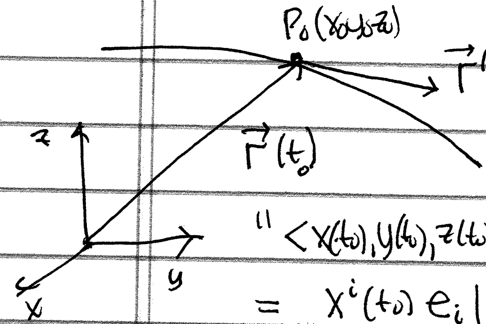
no arrow derivative operator

$$df|_{P_0}(\vec{X}) = X f$$

difference vector = problem for curved spaces (no difference vectors)

$\boxed{5.1\text{-}2.}$

solution: tangent vectors to curves



$$\vec r\,'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0)\rangle = x'(t_0)\vec e_1|_{P_0} + y'(t_0)\vec e_2|_{P_0} + z'(t_0)\vec e_3|_{P_0}$$

$$\vec r(t_0) \; " \; \langle x(t_0), y(t_0), z(t_0)\rangle = x^i(t_0)\, e_i|_{P_0}$$

tangent vectors at P_0 = tangent vectors to all curves through P_0

CHAIN RULE:

$$\frac{d}{dt} f(\vec r(t))\Big|_{t=t_0} = f_x(P_0)\,x'(t_0) + f_y(P_0)\,y'(t_0) + f_z(P_0)\,z'(t_0)$$
$$= x^{i\,\prime}(t_0)\,\frac{\partial f}{\partial x^i}(P_0)$$
$$= x^{i\,\prime}(t_0)\,\frac{\partial}{\partial x^i}\Big|_{P_0} f$$
$$= df|_{P_0}\big(\vec r\,'(t_0)\big)$$

$\boxed{\text{derivative along coordinate lines}}$

x^1:
$$\vec r(t) = \langle x_0^1 + t, x_0^2, x_0^3\rangle$$
$$\vec r\,'(t) = \langle 1, 0, 0\rangle = \vec e_1|_{P_0} = \underbrace{\delta^j_1}_{\text{components}} \vec e_j|_{P_0}$$

$$\underbrace{\frac{d}{dt} x^1(\vec r(t))}_{\text{derivative along } x^1} = dx^1|_{P_0}(\vec r\,'(t_0)) = \delta^j_1 \frac{\partial}{\partial x^j}\Big|_{P_0} x^1$$
$$= \underbrace{\frac{\partial}{\partial x^1}\Big|_{P_0}}_{} x^1$$

corresponds to $\frac{\partial}{\partial x^1}\Big|_{P_0}$ obvious, right?

this is how we defined partial derivatives!

$$\frac{\partial}{\partial x} f(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{t\to 0} \frac{f(x+t,y) - f(x,y)}{t}$$
$$= \frac{d}{dt} f(x+t,y)\Big|_{t=0}$$

translation along coord line.

$y^i(x)$ change of ind. variables
chain rule

$$\underbrace{\frac{d}{dt} f(\vec{r}(t))\Big|_{t=t_0}}_{\substack{\text{does not depend on}\\ \text{how expressed}\\ = \text{"invariant"}}} = \underbrace{\overbrace{x^{i\prime}(t_0)}^{\text{old components}} \frac{\partial}{\partial x^i}\Big|_{P_0}}_{\substack{\text{corresponds to}\\ \text{this invariant}\\ \text{derivative at } P_0\\ \text{(good)}}} f = \overbrace{\underbrace{x^{i\prime}(t_0) \frac{\partial y^j}{\partial x^i}\Big|_{P_0}}_{\text{new components}}} \frac{\partial}{\partial y^j}\Big|_{P_0} f$$

"old basis" "new basis"

$x^{i\prime}(t_0)\, \vec{e}_i\big|_{P_0}$ difference vector (bad)

natural identification $\frac{\partial}{\partial x^i}\Big|_{P_0} \leftrightarrow \vec{e}_i\big|_{P_0}$

re-interpret tangent vector
as the corresponding directional derivative

$$\underset{\text{difference vector}}{\vec{X} = x^i \vec{e}_i\big|_{P_0}} \longleftrightarrow \underset{\text{derivative operator}}{X = X^i \frac{\partial}{\partial x^i}\Big|_{P_0}}$$

same components, new "placeholders" = basis vectors

let $\underset{\text{no arrow}}{e_i}\big|_{P_0} = \frac{\partial}{\partial x^i}\Big|_{P_0}$ basis of tangent space. tangent vectors to coordinate lines

re-define dual basis

$df|_{P_0}(\vec{X}) = X^i \frac{\partial}{\partial x^i}\Big|_{P_0} f = X f$ $\quad X$ tangent vector at P_0

$\Downarrow$

$df|_{P_0}(X) \equiv X f$ new covector at P_0

$$dx^i|_{p_0}(X) = X\, x^i = X^j \underbrace{\frac{\partial}{\partial x^j}\Big|_{p_0} x^i}_{\delta^i{}_j} = X^i$$

$\underbrace{dx^i|_{p_0}}$ dual basis picks out components

$$dx^i|_{p_0}(e_j|_{p_0}) = \delta^i{}_j$$

$$e_j|_{p_0} = \frac{\partial}{\partial x^j}\Big|_{p_0}$$

tangent vectors at p_0: $T\mathbb{R}^3{}_{p_0} = V$

dualspace $(T\mathbb{R}^3{}_{p_0})^* = V^*$

} tensor algebra over V for each p_0

vector <u>field</u> $X = \underbrace{X^i(x)}_{\text{functions on } \mathbb{R}^3} \frac{\partial}{\partial x^i}$ — smooth choice of tangent vector in each $T\mathbb{R}^3{}_p$

tensor <u>field</u> $T = \underbrace{T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}}_{\text{functions on } \mathbb{R}^3} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}$

dotproduct → $G = \delta_{ij}\, dx^i \otimes dx^j$ inner product tensor field = "metric"

$I = \delta^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j = \frac{\partial}{\partial x^i} \otimes dx^i$ identity tensor field

<u>Cartesian coordinate transformations</u> on $\mathbb{R}^n$ induce one on each tangent space

$x^{i'} = A^i{}_j x^j$, $x^i = A^{-1\,i}{}_j x^{j'}$ ⟵ $\underline{A} = (A^i{}_j)$ = constant matrix

$dx^{i'} = A^i{}_j\, dx^j$ $\quad dx^i = A^{-1\,i}{}_j\, dx^{j'}$

$\frac{\partial}{\partial x^{j'}} = \frac{\partial x^i}{\partial x^{j'}} \frac{\partial}{\partial x^i} = A^{-1\,i}{}_j \frac{\partial}{\partial x^i}$ $\quad \frac{\partial}{\partial x^i} = \frac{\partial x^{j'}}{\partial x^i} \frac{\partial}{\partial x^{j'}} = A^j{}_i \frac{\partial}{\partial x^{j'}}$

} change of basis

$$T = T^{i\ldots}{}_{j\ldots} \frac{\partial}{\partial x^i} \otimes \cdots \otimes dx^j \otimes \cdots = T^{i'\ldots}{}_{j'\ldots} \frac{\partial}{\partial x^{i'}} \otimes \cdots \otimes dx^{j'} \otimes \cdots$$

$$T^{i'\ldots}{}_{j'\ldots} = A^i{}_m \cdots A^{-1\,n}{}_j \cdots T^{m\ldots}{}_{n\ldots}$$ component transformation

differential of a function

$$df = \frac{\partial f}{\partial x^i} dx^i = \text{covector field} = \text{"1-form"}$$
$$= \underbrace{f_{,i}}_{\text{components}} dx^i$$

(comma = reminder of derivative)

gradient vector field

$$\vec{\nabla} f = (\vec{\nabla} f)^j \frac{\partial}{\partial x^j} = \delta^{ij} f_{,j} \frac{\partial}{\partial x^i} = \text{index raised differential}$$

2-vector field: $F = \frac{1}{2} F^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$

2-covector field = 2-form: $H = \frac{1}{2} H_{ij}\, dx^i \wedge dx^j$

p-forms = differential form's

$$\eta = \underbrace{\eta_{123}}_{1 \text{ in standard basis}} dx^1 \wedge dx^2 \wedge dx^3$$

unit 3-form = local determinant

$$\eta(X, Y, Z) = \det(X, Y, Z)$$

but in new sense of tangent vectors

$\boxed{\text{everything extends to } \mathbb{R}^n \text{ and its tensor field algebra}}$

and to "curvilinear coord systems"

change of basis fields

$$\frac{\partial}{\partial x^i} = \underbrace{\frac{\partial x^{j'}}{\partial x^i}} \frac{\partial}{\partial x^{j'}} \longleftrightarrow \frac{\partial}{\partial x^{i'}} = \underbrace{\frac{\partial x^j}{\partial x^{i'}}} \frac{\partial}{\partial x^j}$$

$\underline{J} = (J^j{}_i)$ Jacobian matrix

$\underline{J}^{-1} = (J^{-1\,j}{}_i)$ inverse Jacobian = Jacobian of inverse transformation

"frame" = field of bases

"reference frame" for measuring local "geometry" and to general noncoordinate frames!

$$e_{i'} = A^{-1\,j}{}_{i} \frac{\partial}{\partial x^j} \qquad \omega^{i'} = A^{i'}{}_{j}\, dx^j$$
$$= e^{j}{}_{i'} \frac{\partial}{\partial x^j} \qquad = \omega^{i'}{}_{j}\, dx^j$$

$$(e^{j}{}_{i'}) \leftrightarrow (\omega^{i'}{}_{j})$$ inverse matrices

most useful such frames are orthonormal
since can directly interpret components
in terms of local lengths and angles like we do
in $\mathbb{R}^n$ itself ($\mathbb{R}^3$!)
in Cartesian coordinates.

as move around in the space,
such frames rotate, rates of change define
antisymmetric matrices

$\updownarrow$

vector fields

$\downarrow$

$\boxed{5.3}$ flowlines of
vector fields.

5.3-4.

1

vector field on $\mathbb{R}^n$: identified with derivative operator

$\vec{X} = \underset{\sim}{X^i} \vec{e}_i \longleftrightarrow X = X^i e_i \equiv X^i \frac{\partial}{\partial x^i} = X^i \partial_i$

component functions

standard basis at each point

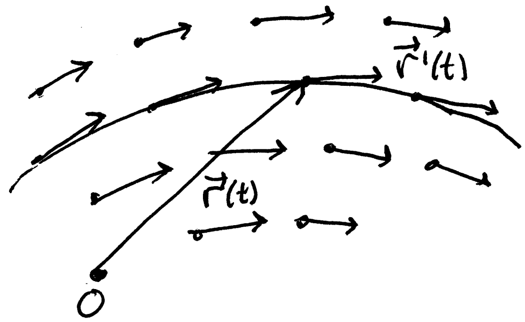
associated Cartesian coords

covector field $\sigma = \sigma_i \omega^i \longleftrightarrow \sigma = \sigma_i dx^i$

dual basis at each pt

differential: $df(\vec{X}) = X^i \partial_i f \longleftrightarrow df(X) = X^i \partial_i f$

Geometry of vector fields? $\boxed{\text{Flow lines}}$



vector field: $\begin{cases} \vec{\xi} = \xi^i e_i \\ \xi = \xi^i(x)\partial_i \end{cases}$

flow line: $\vec{r}(t)$ defined by

tangent $\vec{r}\,'(t) = \vec{\xi}(\vec{r}(t)) =$ value of field at $\vec{r}(t)$

$$\frac{dx^i(t)}{dt} = \xi^i(x(t))$$

add initial condition: $x^i(0) = x_0^i$ starting point on flow line

$\Rightarrow$ initial value problem for first order system of ODEs.

solution: $x^i = x^i(t, x_0)$ = coords of point on flow line starting at x_0^i

not easy to solve in general if $\xi^i = \xi^i(x)$ nonlinear function of position.

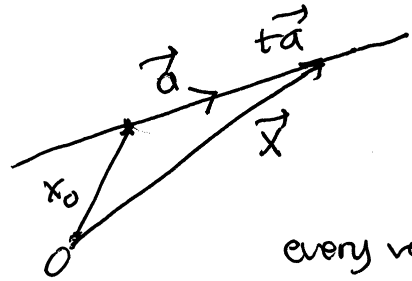
special cases:

1) constant vector field $\xi = \xi^i \partial_i = a^i \partial_i$ a^i constants.

$$\frac{dx^i}{dt} = a^i \rightarrow x^i = \int a^i dt = a^i t + c^i = a^i t + x_0^i$$

$$x^i(0) = 0 + c^i = x_0^i$$

$$\vec{x} = t\vec{a} + \vec{x}_0$$



flowline = straight line through $\vec{x}_0$

these are called $\boxed{\text{translations of space}}$:

every vector $\underline{x_0}$ has the same vector $t\underline{a}$ added to it.

$\boxed{5.3\text{-}4.}$

2

2) linear vector field: $\xi = \xi^i \partial_i = A^i{}_j x^j \partial_i$

$$\frac{dx^i}{dt} = A^i{}_j x^j \qquad \boxed{\frac{d\vec{x}}{dt} = \underline{A}\vec{x}} \qquad \text{eigenvector soln technique!}$$

$$\underline{B} = \langle \underline{b}_1 | \cdots | \underline{b}_n \rangle \qquad \underline{A}\,\underline{b}_i = \lambda_i \underline{b}_i$$

no sum

$$\underline{x} = B\underline{y}\,, \quad \underline{y} = B^{-1}\underline{x} \qquad \underline{A}(y^i \underline{b}_i) = y^i \underline{A}\,\underline{b}_i = (\lambda_i y^i)\,\underline{b}_i$$

$$\underline{x}' = A\underline{x} \to B^{-1}[(By)' = ABy]$$

$$\underline{A}_B = B^{-1}AB = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

in new basis mult by A just scales each basis vector by eigenvalue

$$\underline{B^{-1}B}\,\underline{y}' = \underbrace{B^{-1}AB}\,y$$

$$\underline{y}' = \underline{A}_B \underline{y}$$

$$y^{i\prime} = A_B{}^i{}_j y^i = \lambda_i y^i \quad (\text{no sum})$$

$$y^i = c^i e^{t\lambda_i} \longrightarrow \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} = \underbrace{\begin{bmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{bmatrix}}_{e^{t\underline{A}_B}} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\underline{x} = \underline{B}\,\underline{y} = \underline{B}\,\underline{e}^{tA_B}\,\underline{C}$$

$$\underline{x}(0) = \underline{B}\,\underline{C} = \underline{x}_0$$

$$\underline{C} = \underline{B}^{-1}\underline{x}_0 \quad \text{new coords of } \underline{x}_0$$

$$\underline{x} = B e^{t A_B} B^{-1} \underline{x}_0$$

$$= \underline{B}\left(\sum_{n=0}^{\infty} t^n \frac{\underline{A}_B^n}{n!}\right) B^{-1} \underline{x}_0$$

$$= \left(\sum_{n=0}^{\infty} \underline{\frac{t^n}{n!} B\,A_B^n\,B^{-1}}\right) \underline{x}_0 \qquad = \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \underline{A}^n\right) \underline{x}_0$$

$$B A_B B^{-1} B A_B B^{-1} \cdots B A_B B^{-1}$$

$$\underbrace{(B A_B B^{-1})^n}_{\underline{A}} \qquad = e^{t\underline{A}}\,\underline{x}_0$$

$$\boxed{\underline{x} = e^{t\underline{A}}\,\underline{x}_0} \quad \text{linear transformations of space.}$$

$$\frac{d}{dt} e^{tA} = \underline{A} e^{t\underline{A}} = e^{t\underline{A}} \underline{A}$$

check soln

$$\frac{d\underline{x}}{dt} = A \underbrace{e^{t\underline{A}} x_0}_{\underline{x}} = A\underline{x}$$

$\boxed{5.3\text{-}4.}$ 3

<u>general vector field</u>: $\xi = \xi^j(x)\,\partial_j$

notice $\xi x^i = \xi^j \partial_j x^i = \xi^j \delta^i{}_j = \xi^i$

differentiate coordinates — get components.

of course: $\xi x^i = dx^i(\xi) = \xi^i$ by construction.

flow lines: $\dfrac{dx^i(t)}{dt} = \xi^i(x(t)) = (\xi x^i)\big|_{x(t)}$

↓ differentiate $\dfrac{d}{dt}$

$$\frac{d^2x^i(t)}{dt^2} = \underbrace{\frac{dx^k(t)}{dt}\frac{\partial \xi^i(x(t))}{\partial x^k}}_{\text{chain rule to diff } \xi^i \text{ along } x^i(t)}$$

$$= \xi^k(x(t))\frac{\partial \xi^i(x(t))}{\partial x^k}$$

$$= \xi\,(\,\xi\, x^i)\big|_{x(t)} = (\xi^2 x^i)\big|_{x(t)}$$

...

$$\frac{d^n x^i(t)}{dt^n} = (\xi^n x^i)\big|_{x(t)}$$

Taylor series representation for solution $x^i(t)$:

$$x^i(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \underbrace{(x^i)^{(k)}(0)}_{(\xi^k x^i)|_{x(0)\equiv x_0}}$$

$$= \left(\sum_{k=0}^{\infty} \frac{t^k \xi^k}{k!}\right) x^i \Big|_{x=x_0} = e^{t\xi} x^i \Big|_{x=x_0}$$

previous special cases:

1) constant: $\xi^i = a^i$ $\quad \xi x^i = a^i, \quad \xi^2 x^i = 0 = \xi^n, n>1$

$e^{t\xi} x^i = x^i + t a^i + 0$

2) linear $\quad \xi^i = A^i{}_j x^j \partial_i, \quad \xi x^i = A^k{}_j x^j \partial_k x^i = A^k{}_j x^j \delta^i{}_k$

$= A^i{}_j x^j = [\underline{A}\,\underline{x}]^i$

$\xi^2 x^i = [\underline{A}^2 \underline{x}]^i$ etc.

$e^{t\xi} x^i = [e^{t\underline{A}}\underline{x}]^i$ reduces to matrix exponential

$$\boxed{e^{t\xi} x^i = \text{coords of point translated } t \text{ units along flow starting at } x^i}$$

vectorfield derivative interpretation is useful here!

$\boxed{5.3\text{-}4}$

4

1) constant vector fields generate translations : $\mathbb{R}^3$

basis : $p_i = \partial_i = \nabla_i$ ↑ linear momentum operators

translation along ith axis

$$(T\vec{x})^i = x^i + a^i = e^{a^j p_j}\, x^i$$

$$a^j p_j = \vec{a}\cdot\vec{\nabla}$$

2) "angular momentum operators" generate rotations of $\mathbb{R}^3$

$$\underline{L}_1 = \begin{bmatrix} 0&0&0\\ 0&0&-1\\ 0&1&0 \end{bmatrix},\quad \underline{L}_2 = \begin{bmatrix} 0&0&1\\ 0&0&0\\ -1&0&0 \end{bmatrix},\quad \underline{L}_3 = \begin{bmatrix} 0&-1&0\\ 1&0&0\\ 0&0&0 \end{bmatrix}$$

$(L_3)^1{}_2 = -1 = \epsilon_{132}$

$$[\underline{L}_i]^j{}_k = \epsilon_{jik} \qquad [\underline{L}_i, \underline{L}_j] = \epsilon_{ijk}\underline{L}_k$$

vector fields : $L_i = [\underline{L}_i]^j{}_k x^k \partial_j = \epsilon_{jik} x^k \partial_j = \epsilon_{ikj} x^k \partial_j$
$= (\vec{r}\times\vec{p})_i$

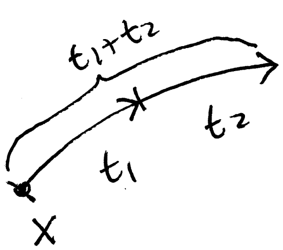
rotation about ith axis

$$R(\theta,\hat{n})\, x^i = e^{\theta \hat{n}^j L_j}\, x^i$$

rotation about axis $\hat{n}$ by angle θ.

Any vector field ξ generates a 1-parameter group of transformations
= translations along its flow lines ← t

$$\frac{dx^i}{dt} = \xi^i(x) \longrightarrow x^i = f^i(x,t), \quad f^i(x,0) = x^i \text{ initial condition}$$
$$= e^{t\xi} x^i$$



flow along by t_1, then continue by t_2
= flow by t_1+t_2 (Abelian)

$$f(x,t_1+t_2)^i = e^{t_1\xi}e^{t_2\xi}x^i$$
$$= e^{(t_1+t_2)\xi}x^i$$

$\boxed{5.3\text{-}4.}$

5

Matrix transformation groups

$[\underline{E}_a, \underline{E}_b] = C^c{}_{ab}\underline{E}_c$ $\quad a,b,c = 1,\ldots,r$ any r-dimensional Lie subalgebra of $n\times n$ matrices.

$\downarrow$

$E_a = E^i{}_{a}{}_j x^j \partial_i$ generating vector fields on $\mathbb{R}^n$

Exercise. $[E_a, E_b] \equiv E_a E_b - E_b E_a = -C^c{}_{ab} E_c$

$= [-E_a, -E_b]$

$\{-E_a\}$ have same commutation relations as $\{\underline{E}_a\}$

Lie bracket of vector fields

$[X, Y] = X^i \partial_i Y^j \partial_j - Y^j \partial_j X^i \partial_i$?? act on functions, right?

$$[X,Y]f = X^i \partial_i (Y^j \partial_j f) - Y^j \partial_j (X^i \partial_i f)$$

$$= X^i (\partial_i Y^j)\partial_j f + X^i Y^j \partial_i \partial_j f$$

$$- Y^j (\partial_j X^i)\partial_i f - Y^j X^i \partial_j \partial_i f$$

$\partial_i \partial_j f = \partial_j \partial_i f$!

$$= \underbrace{(X^i \partial_i Y^j - Y^i \partial_i X^j)}_{[X,Y]^j} f$$

$[X,Y]^j$ new vector field $:= XY^i - YX^i$

$$[e^{\theta^a \underline{E}_a} \underline{x}]^i = e^{\theta^a E_a} x^i$$

$\underline{E}_a$: Lie algebra of matrix group

E_a: Lie algebra of corresponding transformation group on $\mathbb{R}^n$

not surprising that commutators on both sides correspond nicely

<u>Fact</u> Any set of vector fields $\{E_a\}$ whose Lie brackets close:

$[E_a, E_b] = C^c{}_{ab} E_c$ (Lie brackets belong to their span)

generate a group of transformations!

recall for matrix group: $e^{\underline{A}} e^{\underline{B}} = e^{\underline{A} + \underline{B} + \underbrace{\cdots}}$

higher order terms = nested commutators all known if all commutators lie in Lie algebra

exponential of vector field combination:

$x^i \to f^i(x, \theta) = e^{\theta^a E_a} x^i$

$$e^{\theta_1{}^a E_a} e^{\theta_2{}^b E_b} = e^{\theta_1{}^a E_a + \theta_2{}^a E_a + \underbrace{\cdots}}$$

same story for vector fields

Cute, but we only need translations & matrix group transformations. But it helps drive the nail home in the importance of vector fields as derivative operators

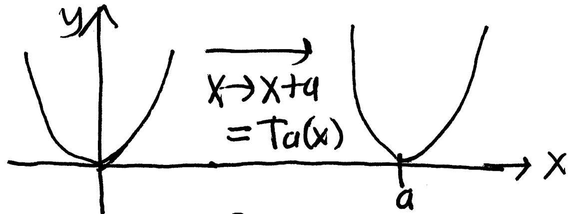
but derivatives like to act on functions, right?

5.3-4

7

How do functions behave under the flow of vector fields?

calc 1: translate graphs



$y = f(x) = x^2$ $\qquad$ $y = f(x-a) = (x-a)^2 = f \circ T_a^{-1}(x)$

$$f \to f \circ T_a^{-1}$$

moving the graph along with the translation composes the function with the inverse.

For any transformation group on $\mathbb{R}^n$,

same order

points: $x \to Tx$ $\qquad$ $x \to T_2(T_1 x) = T_2 \circ T_1 x$

functions: $F \to F \circ T^{-1}$ $\qquad$ $F \to F \circ T_1^{-1} \circ T_2^{-1} = F \circ (T_2 T_1)^{-1}$

This extends action of group from $\mathbb{R}^n$ to $\underbrace{\text{functions on } \mathbb{R}^n}$

"dragging along of functions"

∞-dimensional vector space

$(c_1 F_1 + c_2 F_2)(x) = c_1 F_1(x) + c_2 F_2(x)$

linear combinations defined.

Action of matrix group on a vector space = representation

$x^i \to f^i(x,t) = e^{t\xi} x^i$, $f^i(x,0) = x^i$ $\quad$ flow of vector field

$$\frac{dx^k}{dt} = \xi^k$$

$$F(f(x,t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\frac{d}{dt}(F \circ f)(x,t)\Big|_{t=0}}$$

CHAIN RULE $\quad \frac{d}{dt} F(f(x,t)) = \frac{\partial F}{\partial x^k}(f(x,t)) \frac{dx^k}{dt} = (\xi^k \partial_k F)(f(x,t))$

$$\frac{d}{dt} F(f(x,t))\Big|_{t=0} = (\xi F)(x) \quad \text{etc}$$

$$= \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \xi^n\right) F(x) = (e^{t\xi} F)(x)$$

dragged along function: $\boxed{F \circ f^{-1}(x,t) = e^{-t\xi} F(x)}$

explains sign in Lie bracket compared to matrix commutators

Why do we care?

Functions invariant under a group (like rotations) are nice, but we want to quantify how functions deviate from invariance.

invariance: $e^{-t\xi} F(x) = F(x) \Leftrightarrow \xi F = 0$

function is constant along flow lines

$\boxed{\xi F \neq 0 \text{ describes failure to be invariant.}}$

→ How do we understand matrix group action on $\mathbb{R}^n$?

$\underline{x} \to \underline{A}\underline{x}$ → find basis of eigenvectors

decompose $\mathbb{R}^n$ into direct sum of eigen spaces

$\underline{A}\underline{b}_i = \lambda_i \underline{b}_i$

$\underline{x} = y^i \underline{b}_i = \underline{B}\,\underline{y}$

$\underline{A}\underline{x} = \underline{A}(y^i \underline{b}_i) = y^i (\underline{A}\underline{b}_i) = (\lambda_i y^i)\vec{b}_i$

matrix multiplication reduces to scalar multiplication

→ matrix group action on ∞-dim vector space of functions over $\mathbb{R}^n$?

find basis of "eigenvectors" of generators

rotations: $\underline{L}_i \to L_i$ represent $\underline{L}_i$ on functions

but $[L_i, L_j] f = \epsilon_{ijk} L_k f$ $\quad$ $\begin{cases} L_1 f = \lambda_1 f \\ L_2 f = \lambda_2 f \end{cases}$ $\quad L_3 f = \lambda_3 f$?

if $\overset{\shortparallel}{(\lambda_i \lambda_j - \lambda_j \lambda_i) f}$ $= 0$ $\qquad \epsilon_{ijk} \lambda_k f \neq 0$

cannot simultaneously diagonalize more than one of L_1, L_2, L_3

but $L^2 = L_1^2 + L_2^2 + L_3^2$ satisfies $[L^2, L_i] = 0$

can get eigenfunctions of (L_3, L^2)

L^2: $-\ell(\ell+1)$

L_3: $m = -\ell, -\ell+1, \ldots 0, 1, \ldots \ell-1, \ell \quad (2\ell+1)$

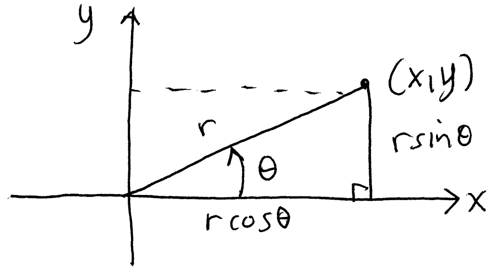
→ spherical harmonics, bigger ℓ = smaller wavelength structure.

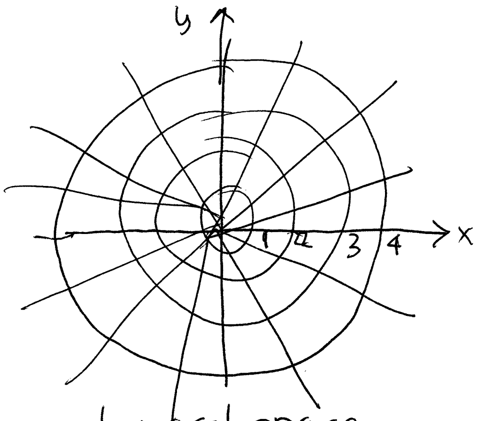
NOT YET READY — MUST WAIT FOR DETAILS

5.5–5.7

1

polar coordinates





physical space

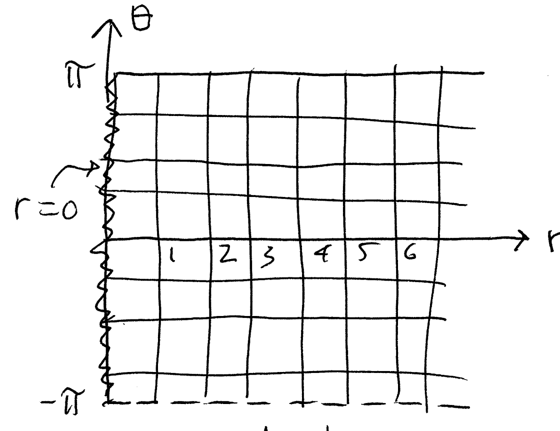
Ψ

parametrization

inverse maps

Φ

coordinatization



coordinate space

$\langle x,y\rangle = \vec{\Psi}(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$

$r \geq 0$

$-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$) as convenient

1-1 map except at $r=0$

$\langle r,\theta\rangle = \Phi(x,y) = (\sqrt{x^2+y^2}, \Theta(x,y))$

$\Theta \equiv \arctan\frac{y}{x} + \begin{cases} 0, & \text{I, IV quads} \\ \pi, & \text{II quad} \\ -\pi, & \text{III quad} \end{cases}$

not defined at $(0,0)$

"coordinate singularity"

point $(0,0)$ $\leftrightarrow$ line segment $r=0$, $\theta=-\pi..\pi$

map breaks down

$\Psi = \Phi^{-1}$, $\Phi = \Psi^{-1}$ once domains restricted

Ψ parametrizes points of space (like parametrized curves, surfaces)

Φ assigns coordinate values to points of space

Ψ transfers Cartesian grid on coordinate space to polar coord grid on physical space

"coordinate lines"

$(x^1, x^2) = (x,y) \longleftrightarrow (x^{1'}, x^{2'}) = (r,\theta)$

transformation of coordinates

$\{e_1, e_2\} = \{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}\}$
$= \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$

$\{e_{1'}, e_{2'}\} = \{\frac{\partial}{\partial x^{1'}}, \frac{\partial}{\partial x^{2'}}\}$
$= \{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\}$

$\boxed{5.5-5.7}$

2

tangent vectors to coord lines $\longrightarrow$ coordinate derivatives = coordinate frame

$$\vec{\Psi} = \langle x, y\rangle = \langle r\cos\theta, r\sin\theta\rangle$$
$$= \langle x^i\rangle$$

$$\left\{\begin{aligned} \vec{e}_{1'} &= \frac{\partial\vec{\Psi}}{\partial r} = \langle\cos\theta, \sin\theta\rangle = \frac{\partial\langle x^i\rangle}{\partial r} \\ &= \left\langle\frac{x}{r}, \frac{y}{r}\right\rangle \\ \vec{e}_{2'} &= \frac{\partial\vec{\Psi}}{\partial\theta} = \langle -r\sin\theta, r\cos\theta\rangle = \frac{\partial\langle x^i\rangle}{\partial\theta} \\ &= \langle -y, +x\rangle \end{aligned}\right.$$

$$\left\{\begin{aligned} e_{1'} &= \frac{x}{r}\frac{\partial}{\partial x} + \frac{y}{r}\frac{\partial}{\partial y} = \frac{\partial x^i}{\partial r}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial r} \\ e_{2'} &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \frac{\partial x^i}{\partial\theta}\frac{\partial}{\partial x^i} = \frac{\partial}{\partial\theta} \end{aligned}\right.$$

just chainrule

$$\underline{G_{i'j'} = \vec{e}_{i'}\cdot\vec{e}_{j'}}$$
$$= e_{i'}\cdot e_{j'}$$

$$G_{1'1'} = G_{rr} = \vec{e}_{1'}\cdot\vec{e}_{1'} = \cos^2\theta + \sin^2\theta = 1$$
$$G_{2'2'} = G_{\theta\theta} = \vec{e}_{2'}\cdot\vec{e}_{2'} = r^2\sin^2\theta + r^2\cos^2\theta = r^2$$
$$G_{1'2'} = G_{r\theta} = \vec{e}_{1'}\cdot\vec{e}_{2'} = -r\cos\theta\sin\theta + r\cos\theta\sin\theta = 0$$

$$G = \delta_{ij}\, dx^i\otimes dx^j = G_{i'j'}\, dx^{i'}\otimes dx^{j'} = 1\, dr\otimes dr + r^2\, d\theta\otimes d\theta$$

orthogonal coordinates

<u>normalize</u>

$$\hat{e}_{1'} = e_{1'} = \frac{\partial}{\partial r}$$
$$\hat{e}_{2'} = \frac{1}{r}e_{2'} = \frac{1}{r}\frac{\partial}{\partial\theta}$$

(divideby sqrt $G_{i'i'}$)

$$\hat{\omega}^{1'} = dr$$
$$\hat{\omega}^{2'} = r\,d\theta$$

(multiply by sqrt $G_{i'i'}$)

dual frame: $\{dr, d\theta\} = \{dx^{1'}, dx^{2'}\} = \{\omega^{1'}, \omega^{2'}\}$

$$\left\{\begin{aligned} dr\left(\frac{\partial}{\partial r}\right) &= \frac{\partial}{\partial r} r = 1 \\ d\theta\left(\frac{\partial}{\partial\theta}\right) &= \frac{\partial}{\partial\theta}\theta = 1 \text{ etc} \end{aligned}\right.$$

$$\boxed{G = \hat{\omega}^{1'}\otimes\hat{\omega}^{1'} + \hat{\omega}^{2'}\otimes\hat{\omega}^{2'}}$$

orthonomal frame.

$\boxed{5.5\text{-}5.7}$

3

$e_{i'} = e_j B^j{}_{i'}$

$$\begin{array}{ccc} \text{new} & & \text{old} \\ (e_r\ e_\theta) & = & [e_1\ e_2] \end{array} \underbrace{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}}_{\underline{B} = \underline{A}^{-1}}$$

columns = new basis

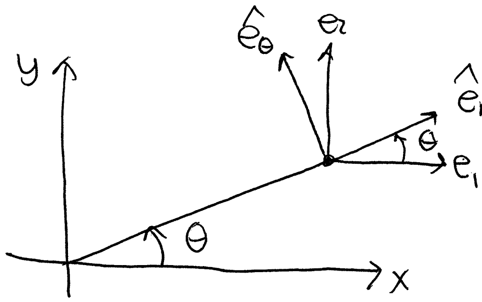
$$\underline{B} = \begin{bmatrix} x/r & -y \\ y/r & x \end{bmatrix}$$

$\hat{e}_{i'} = e_j \mathcal{B}^j{}_i$

$$[\hat{e}_r\ \hat{e}_\theta] = [e_1\ e_2] \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\mathcal{B} = \underline{\mathcal{A}}^{-1}}$$

= active rotation of plane

same as rotation of whole space



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \rightarrow \overset{\text{old}}{\begin{bmatrix} dx \\ dy \end{bmatrix}} = \begin{bmatrix} dr\cos\theta - r\sin\theta\, d\theta \\ dr\sin\theta + r\cos\theta\, d\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}}_{\underline{B}} \overset{\text{new}}{\begin{bmatrix} dr \\ d\theta \end{bmatrix}}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\underline{\mathcal{B}}} \begin{bmatrix} dr \\ r\,d\theta \end{bmatrix}$$

$$\begin{array}{ccc} \text{old} & & \text{new} \\ [e_1\ e_2] & = & [e_r\ e_\theta] \end{array} \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}}_{\underline{B}^{-1} = \underline{A}} = [\hat{e}_r\ \hat{e}_\theta] \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{\underline{\mathcal{B}}^{-1} = \underline{\mathcal{A}}}$$

$$\underline{B}^{-1} = \underline{A} = \begin{bmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{bmatrix}$$

$$\overset{\text{new}}{\begin{bmatrix} dr \\ d\theta \end{bmatrix}} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix} \overset{\text{old}}{\begin{bmatrix} dx \\ dy \end{bmatrix}}$$

$$\begin{bmatrix} dr \\ r\,d\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$\boxed{5.5-5.7}$

4.

Inverse Jacobian matrices transform frames old $\Leftrightarrow$ new

$$\frac{\partial x^{i'}}{\partial x^{j}}(x) = \frac{\partial x^{i'}}{\partial x^{j}}(\underbrace{x(x')}_{})$$

re-express in terms of other coords

$$\frac{\partial x^{i}}{\partial x^{j'}}(x') = \frac{\partial x^{i}}{\partial x^{j'}}(x'(x))$$

can use matrices to transform components or just re-express:

$$X = X^i(x)\frac{\partial}{\partial x^i} = \underbrace{X^i(x)\frac{\partial x^{j'}}{\partial x^i}(x)}_{}\frac{\partial}{\partial x^{j'}} = \underbrace{X^{j'}(x')}_{}\frac{\partial}{\partial x^{j'}}$$ (1) chain rule

$$= X^i(x(x'))\frac{\partial x^{j'}}{\partial x^i}(x(x')) = \text{re-express (2)}$$

$$\sigma = \sigma_i(x)dx^i = \underbrace{\sigma_i(x)\frac{\partial x^i}{\partial x^{j'}}(x)}_{}dx^{j'} = \underbrace{\sigma_{j'}(x')}_{}dx^{j'}$$ (1) differential

$$= \sigma_i(x(x'))\frac{\partial x^i}{\partial x^{j'}}(x') = \text{re-express (2)}$$

4 different Jacobian matrix expressions for the different contexts:
matrix & inverse expressed in old & new coords.

(1) $$X^{i'}(x') = \underbrace{\frac{\partial x^{i'}}{\partial x^j}}_{\text{index transformation}}(x(x'))\,X^j(x(x'))$$

transformation of functional dependence.

(2) $X^i(x) = \ldots$

(3) $\sigma_{i'}(x') = \ldots$

(4) $\sigma^i(x) = \ldots$

ALL NATURAL "re-expression" of derivatives & differentials

$\boxed{5.5\text{-}5.7}$

5

examples

$$\xi = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\,,\quad \sigma = \xi^{b} = y\,dx + x\,dy$$

$$\sigma = \xi^{b} = [y\ \ x]\begin{bmatrix} dx \\ dy \end{bmatrix} = [r\sin\theta\ \ r\cos\theta]\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$= [\underbrace{r\sin\theta\cos\theta + r\cos\theta\sin\theta}_{r\sin 2\theta}\quad \underbrace{-r^2\sin^2\theta + r^2\cos^2\theta}_{r^2\cos 2\theta}]\begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$= \underbrace{r\sin 2\theta}_{\sigma_r}\,dr + \underbrace{r^2\cos 2\theta}_{\sigma_\theta}\,d\theta$$

coord components

$$= \underbrace{(r\sin 2\theta)}_{\sigma_{\hat r}}\,dr + \underbrace{(r\cos 2\theta)}_{\sigma_{\hat\theta}}\,(r\,d\theta)$$

ON components

$$\xi = [\partial_x\ \partial_y]\begin{bmatrix} y \\ x \end{bmatrix} = [\partial_r\ \partial_\theta]\begin{bmatrix} \cos\theta & +\sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{bmatrix}\begin{bmatrix} r\sin\theta \\ r\cos\theta \end{bmatrix}$$

$$= [\partial_r\ \partial_\theta]\begin{bmatrix} r\cos\theta\sin\theta + r\sin\theta\cos\theta \\ -\cancel{r}\sin^2\theta + \cancel{r}\cos^2\theta \end{bmatrix}$$

$$= \underbrace{r\sin 2\theta}_{\xi^r}\,\partial_r + \underbrace{\cancel{r}\cos 2\theta}_{\xi^\theta}\,\partial_\theta = \underbrace{r\sin 2\theta}_{\xi^{\hat r}}\,\partial_r + \underbrace{r\cos 2\theta}_{\xi^{\hat\theta}}\left(\frac{1}{r}\partial_\theta\right)$$

$$\sigma_r = G_{rr}\xi^r = \xi^r \qquad \sigma_{\hat r} = \xi^{\hat r}$$

$$\sigma_\theta = G_{\theta\theta}\xi^\theta = r^2\xi^\theta \qquad \sigma_{\hat\theta} = \xi^{\hat\theta}$$

all works out nicely.

can repeat in opposite direction

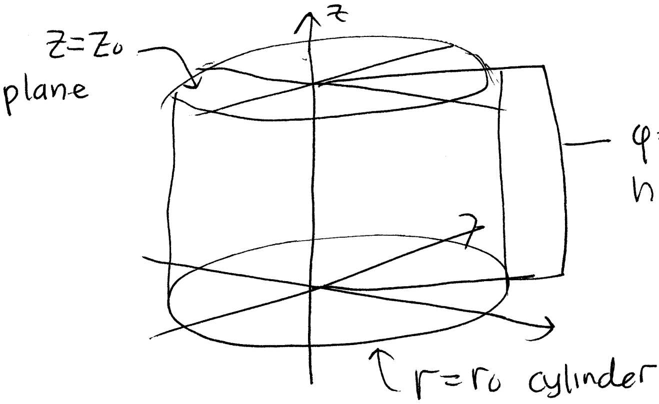
$\boxed{5.5-5.7}$

cylindrical coords on $\mathbb{R}^3$

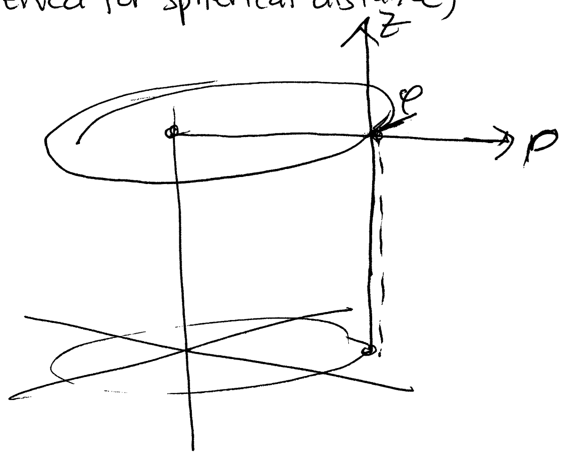
$$\left.\begin{array}{l} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{array}\right\} \qquad z = z$$

change (r,θ) to (ρ,φ) physics convention

↑ (reserved for spherical distance)



coordinate surfaces



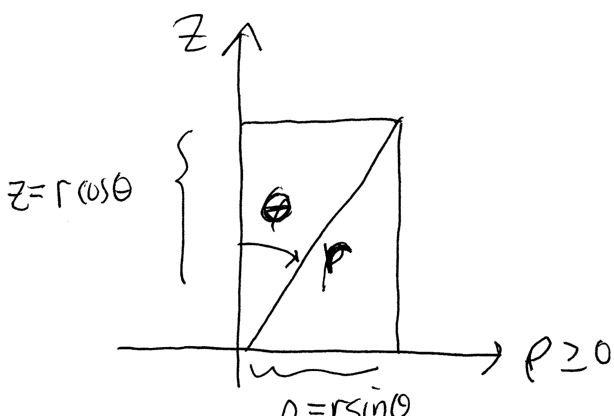
coordinate lines

easy, only replace x, y by ρ, φ keep z.

$\boxed{\text{orthogonal coordinates}}$

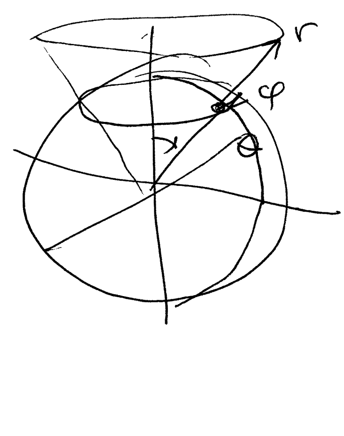
all calculations same as in polar coordinates, plus z.

for rotational symmetry about z-axis
ρ-z halfplane enough!



add polar coords in this halfplane to get spherical coords

"double polar coords"



$$\begin{aligned} x &= \rho\cos\varphi &&= r\sin\theta\cos\varphi \\ y &= \rho\sin\varphi &&= r\sin\theta\sin\varphi \\ z &= z &&= r\cos\theta \end{aligned}$$

$\boxed{5.5\text{–}5.7}$ $\boxed{\text{Projection}}$

7

$$\mathbb{X} = \alpha \hat{u} \qquad \hat{u}\cdot\hat{u} = \epsilon = \pm 1 \quad \text{unit vector}$$

$$\hat{u}\cdot\mathbb{X} = \alpha\,\hat{u}\cdot\hat{u} = \epsilon\alpha \;\rightarrow\; \alpha = \epsilon\,\hat{u}\cdot\hat{\mathbb{X}}$$

$$\mathbb{X} = \underbrace{(\epsilon\,\mathbb{X}\cdot\hat{u})}\,\hat{u}$$

along timelike vector in spacetime $-\mathbb{X}\cdot\hat{u}$ is component along $\hat{u}$

ON Frame

$$\mathbb{X} = \mathbb{X}^i e_i \qquad e_i\cdot e_i = G_{ii} = \pm 1$$

$$\mathbb{X}^i = \frac{\mathbb{X}\cdot e_i}{G_{ii}}$$

$$\mathbb{X} = \sum_{i=1}^{n} e_i \underbrace{\left(\frac{e_i\cdot\mathbb{X}}{G_{ii}}\right)} = e_i\,\omega^i(\mathbb{X}) = (e_i\otimes\omega^i)(\;,\mathbb{X})$$

$$\omega^i(\mathbb{X}) = \left(\frac{e_i}{G_{ii}}\right)\cdot\mathbb{X} \qquad \text{or} \qquad \underbrace{e_i\cdot}\,\mathbb{X} = G_{ii}\,\omega^i(\mathbb{X})$$

$$e_i^{\flat} = G_{ii}\,\omega^i$$

sign of lowering index for timelike component.

on $\mathbb{R}^n$ matrix notation, no signs

$$\underline{x} = y^i \underline{b_i} = \underline{B}\,\underline{y} \qquad b_i\cdot b_j = \delta_{ij}$$

$$y^i = b_i\cdot\underline{x}$$

$$\underline{x} = \sum_{i=1}^{n}\underline{b}_i\,(\underbrace{b_i\cdot\underline{x}}_{=\,\underline{b_i}^T\underline{x}}) = \sum_{i=1}^{n}\underline{b}_i(\underline{b_i}^T\underline{x}) = \Big(\sum_{i=1}^{n}\underbrace{\underline{b_i}\,\underline{b_i}^T}\Big)\,\underline{x}$$

projects along ith basis vector

$$\text{tensor product } b^j_i\, b^k_i \;\longleftrightarrow\; \text{square matrix } \underline{P}_{(i)} = \underline{b}_i\underline{b}_i^T$$

$$b^j_i\,b^k_i = P_{(i)}{}^j{}_k$$

$$\underline{P}_{(i)}{}^2 = \underline{P}_{(i)}, \quad \underline{P}_{(i)}\underline{P}_{(j)} = 0 = \underline{P}_{(j)}\underline{P}_{(i)}$$

ON basis automatically makes orthogonal direct sum decomposition of space

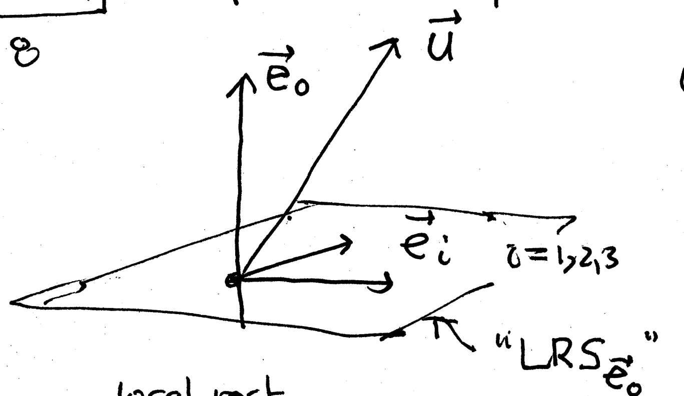
$\boxed{5.5-5.7}$

8

spacetime ON frame $\vec{e}_\alpha \quad \alpha = 0,1,2,3$

$$(\vec{e}_\alpha \cdot \vec{e}_\beta) = (\eta_{\alpha\beta}) = \mathrm{diag}(-1,1,1,1)$$

$$X = \underbrace{X^0 e_0}_{\substack{\text{time}\\ \text{component}\\ \text{(vector)}}} + \underbrace{X^i e_i}_{\substack{\text{spatial}\\ \text{component}\\ \text{(vector)}}}$$



local rest space of $\vec{e}_0$

4-velocity of observer at rest at spatial origin of inertial coords (ON Cartesian coords in M^4)

4-velocity of particle/observer in relative motion

$$\vec{u} = \underbrace{u^0}_{\substack{\equiv \cosh\beta \\ \equiv \gamma}} \vec{e}_0 + u^i e_i \qquad -\underbrace{u^{0\,2}}_{\cosh^2\beta} + \underbrace{\delta_{ij} u^i u^j}_{\sinh^2\beta} = -1$$

$$= u^0 \left(\vec{e}_0 + \frac{u^i}{u^0} e_i\right) = \gamma\,(\vec{e}_0 + \overbrace{v^i e_i}^{\vec{v}\ \text{spatial velocity}})$$

$$\delta_{ij} v^i v^j = \frac{\sinh^2\beta}{\cosh^2\beta} = \tanh^2\beta$$

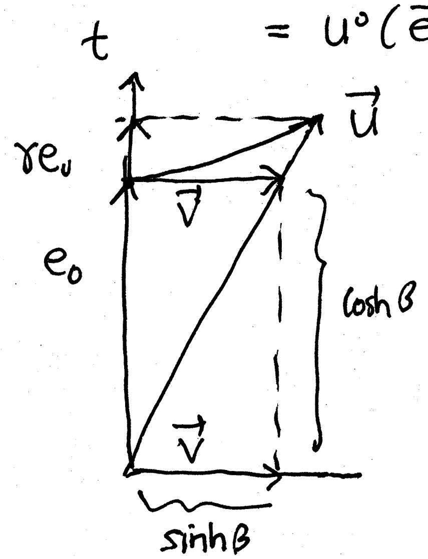
$v = \tanh\beta$ speed

reciprocal slope

$$\gamma = (1 - v^2)^{-1/2}$$

gamma factor

(manipulate hyp identity)



$$\vec{p} = m\vec{u} = m\gamma(e_0 + v^i e_i)$$

$$= \underbrace{(m\gamma)}_{\substack{E\\ \text{energy}}} e_0 + \underbrace{m\gamma v^i}_{} e_i$$

$p^i e_i = \vec{p}$ spatial momentum

projection along 8 orthogonal to a timelike unit vector "measures" spacetime quantity by time plus space quantities

$\boxed{5.5-5.7}$

9

electromagnetic field

$$(F^\alpha{}_\beta) = \begin{bmatrix} 0 & F^0{}_1 & F^0{}_2 & F^0{}_3 \\ F^1{}_0 & 0 & F^1{}_2 & F^1{}_3 \\ F^2{}_0 & F^2{}_1 & 0 & F^2{}_3 \\ F^3{}_0 & F^3{}_1 & F^3{}_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ E^1 & 0 & B^3 & -B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{bmatrix}$$

$$\frac{dp^\alpha}{d\tau} = q F^\alpha{}_\beta U^\beta$$

$\downarrow$

$$U^\alpha = \frac{dx^\alpha}{d\tau} \Rightarrow \frac{dx^0}{d\tau} = \frac{dt}{d\tau} = U^0 = \gamma$$

$$\frac{dp^\alpha}{dt} = q F^\alpha{}_\beta \left(\frac{U^\beta}{\gamma}\right)$$

$$\frac{U^\beta}{\gamma} = (1, V^i)$$

energy $\downarrow$ electric field $\swarrow$

0: $$\frac{dE}{dt} = q\vec{E}\cdot\vec{V}$$

i: $$\frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{V}\times\vec{B}\right)$$