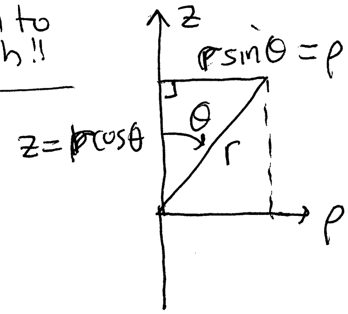
spherical coordinates & their coord frame → ON frame

$$\vec{r} = \langle \underbrace{r\sin\theta}_{\rho}\overset{x}{\cos\varphi},\ \underbrace{r\sin\theta}_{\rho}\overset{y}{\sin\varphi},\ \underbrace{r\cos\theta}_{\overset{z}{z}} \rangle \longleftarrow \text{cyl to sph!!}$$



$$\vec{e}_r = \frac{\partial\vec{r}}{\partial r} = \langle \sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta \rangle$$

$$\vec{e}_\theta = \frac{\partial\vec{r}}{\partial\theta} = \langle r\cos\theta\cos\varphi,\ r\cos\theta\sin\varphi,\ -r\sin\theta \rangle$$

$$\vec{e}_\varphi = \frac{\partial\vec{r}}{\partial\varphi} = \langle -r\sin\theta\sin\varphi,\ +r\sin\theta\cos\varphi,\ 0 \rangle$$

$$\underline{B} = \langle \vec{e}_r | \vec{e}_\theta | \vec{e}_\varphi \rangle = \begin{bmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{bmatrix} \equiv \underline{A}^{-1}$$

orthogonal

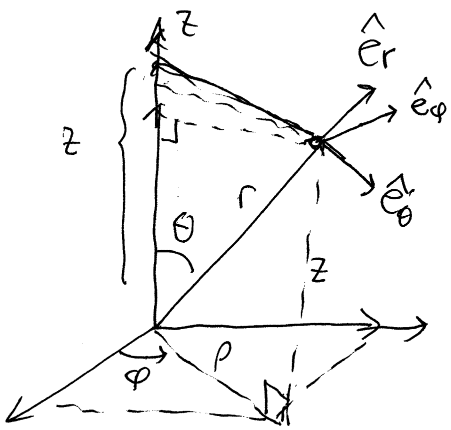
$$\text{length} \quad 1 = G_{rr}^{1/2} \qquad r = G_{\theta\theta}^{1/2} \qquad r\sin\theta = G_{\varphi\varphi}^{1/2}$$

$$\underline{\mathcal{B}} = \langle \hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi \rangle = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} = \mathcal{a}^{-1}$$

orthonormal

orthogonal matrix

unit tangents along coord axes.



$$\vec{e}_{i'} = \vec{e}_j\, B^j{}_i$$

$$\hat{e}_{i'} = \hat{e}_j\, \mathcal{B}^j{}_i$$

new basis labels

$i' = r, \theta, \varphi$

$1' 2' 3'$

↓

derivative operator vector fields

$$e_{i'} = B^j{}_i\, e_j \qquad e_j = \partial/\partial x^j$$

$$e_{\hat{i}'} = \mathcal{B}^j{}_i\, e_j \qquad e_{i'} = \partial/\partial x^{i'}$$

coord frames

↓

ON noncoordinate frame

↓

dual frame $\omega^{\hat{i}'}$

dual frames

$dx^i,\ dx^{i'}$

metric:

$$G = \delta_{ij}\, dx^{\hat{i}} \otimes dx^{\hat{j}}$$

$$= dr\otimes dr + r^2 d\theta\otimes d\theta + r^2\sin^2\theta\, d\varphi\otimes d\varphi$$

$$= \underbrace{\omega^{\hat{r}}\otimes\omega^{\hat{r}}}_{dr} + \underbrace{\omega^{\hat{\theta}}\otimes\omega^{\hat{\theta}}}_{r\,d\theta} + \underbrace{\omega^{\hat{\varphi}}\otimes\omega^{\hat{\varphi}}}_{r\sin\theta\, d\varphi} \longleftarrow$$

derivative of function along vector field

$$Xf = \underset{\text{Cartesian}}{X^i \partial_i f} = \underset{\text{spherical}}{X^{i'} \frac{\partial f}{\partial x^{i'}}} = \underset{\text{ON frame}}{X^{\hat{i}}\, e_{\hat{i}} f} = df(X)$$

$\frac{\partial}{\partial x^i}$ = global constant unit vectors, tensors are constant if Cartesian component functions are constant

How to extend notion of constancy to "moving frames"

<u>covariant derivative</u>

$$\nabla_Y X = \nabla_Y(X^i e_i) = \underbrace{(\nabla_Y X^i)}\, e_i$$

↑ constant Cartesian frame

just ordinary derivative of Cartesian component functions

$$= \nabla_Y(X^{i'} e_{i'}) = (\nabla_Y X^{i'})\, e_{i'} + X^{i'} \underbrace{\nabla_Y e_{i'}}$$

↑ not constant.

contribution from derivative of "moving frame"

new field ↓

$$\nabla_Y e_{i'} = \underbrace{\omega^{j'}{}_{i'}(Y)}\, e_{j'}$$

↑ linear in Y (like $\vec{Y}\cdot\vec{\nabla}$)

↖ just components of $\nabla_Y e_{i'}$ in same frame

$= \cancel{\omega^{j'}(\nabla(Y))}$

$= \omega^{j'}(\nabla_Y e_{i'})$

components of connection

$$\omega'(Y) = (\omega^{j'}{}_{i'}(Y))$$

↑ matrix-valued 1-form

value on vector Y is a matrix, derivative of frame along Y.

$$\omega(Y)^{j'}{}_{i'} = (Y^{k'} e_{k'}) = \underbrace{\omega^{j'}{}_{i'}(e_{k'})}\, Y^{k'}$$

$$\equiv \Gamma^{j'}{}_{k' i'}$$

matrix indices

↑ derivative index

$\boxed{5.8\text{-}5.9,6}$

simpler, more direct!

$$\nabla_{e_{k'}} e_{i'} = \Gamma^{j'}{}_{k'i'} e_{j'}$$

express derivatives of frame vectors along framevectors in terms of the frame vectors

$$\nabla_Y e_{i'} = \nabla_{Y^{k'} e_{k'}} e_{i'} = Y^{k'} \nabla_{e_{k'}} e_{i'}$$
$$= Y^{k'} \Gamma^{j'}{}_{k'i'} e_{j'}$$

linear in vector along which derivative is taken

now apply to X:

$$\nabla_Y X = \nabla_Y (X^{\ell'} e_{\ell'}) = \underbrace{(\nabla_Y X^{\ell'})}_{Y X^{\ell'} = Y^{k'} e_{k'} X^{\ell'}} e_{\ell'} + X^{\ell'} \underbrace{(\nabla_Y e_{\ell'})}_{(\Gamma^{j'}{}_{k'\ell'} Y^{k'} e_{j'})}$$

product rule

derivative of function

$$\underbrace{(\Gamma^{j'}{}_{k'\ell'} Y^{k'} e_{j'}) X^{\ell'}}_{\Gamma^{\ell'}{}_{k'j'} Y^{k'} X^{j'} e_{\ell'}}$$

switch dummy indices

$$= Y^{k'} (\underbrace{e_{k'} X^{\ell'}}_{\text{derivative due to changing components}} + \underbrace{\Gamma^{\ell'}{}_{k'j'} X^{j'}}_{\text{change due to changing frame}}) e_{\ell'}$$

$$= [Y X^{\ell'} + \underbrace{\omega^{\ell'}{}_{j'}(Y)}_{\text{linear transf of tangent space}} X^{j'}] e_{\ell'}$$

= connection 1-form matrix in new frame

(zero in original Cartesian frame since constant)

5.8–5.9, 6

4

$\mathbb{R}^n$ vector fields (multivariable calc)

re-expressed in same frame — old components — derivative of ith vector

$$\vec{e}_{i'} = \vec{e}_j B^j{}_i$$

$$d\vec{e}_{i'} = \underbrace{\vec{e}_j}_{\vec{e}_{k'} B^{-1k}{}_j} dB^j{}_i = \vec{e}_{k'} \underbrace{B^{-1k}{}_j dB^j{}_i}_{\omega^{k'}{}_{i'}}$$

$\omega^{k'}{}_{i'}$ = matrix of 1-forms

$$\nabla_Y \vec{e}_{i'} = d\vec{e}_{i'}(Y) = \vec{e}_{k'} \underbrace{\omega^{k'}{}_{i'}(Y)}$$

$\updownarrow \qquad \updownarrow$

$Yf = df(Y)$

derivative of ith framevector along Y, kth component.

Same relations hold for derivative vector fields

Calculate $\omega^{k'}{}_{i'} = \Gamma^{k'}{}_{j'i'} dx^{j'}$

$\underline{\omega}' = \underline{B}^{-1} d\underline{B}$ easy with Maple.

can go one step further all the way to ON frame

$\omega^{\hat{k}'}{}_{\hat{i}'} = \Gamma^{\hat{k}'}{}_{\hat{j}'\hat{i}'} \omega^{\hat{j}'}$

$\hat{\omega}' = \mathcal{B}^{-1} d\mathcal{B}$ antisymmetric, describes rate of rotation of frame along direction Y when evaluated on Y:

orthogonal matrix

dual = vector

angular velocity differential

$\hat{\omega}'(Y)$

$\boxed{5.8\text{-}5.9,6}$

5

Maple:

$$\omega = \mathcal{B}^{-1} d\mathcal{B} = \begin{array}{ccc} \hat r & \hat\theta & \hat\varphi \\ \end{array}$$
$$\begin{bmatrix} 0 & -d\theta & -\sin\theta\, d\phi \\ d\theta & 0 & -\cos\theta\, d\phi \\ \sin\theta\, d\phi & \cos\theta\, d\phi & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$\langle a_1, a_2, a_3 \rangle$

component dual $\downarrow$

$$\vec\omega = \langle \cos\theta\, d\phi, -\sin\theta\, d\phi, d\theta \rangle$$

vector valued 1-form

but components in ON frame

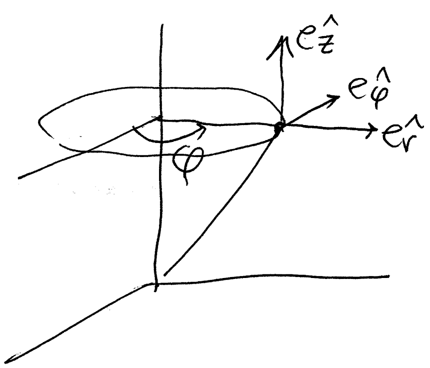
$\downarrow$

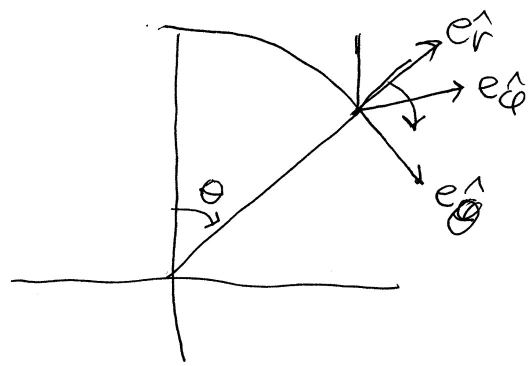
$$\underbrace{\cos\theta\, d\phi\, e_{\hat r} - \sin\theta\, d\phi\, e_{\hat\theta}}_{e_{\hat z}\, d\phi} + d\theta\, e_{\hat\varphi}$$

$e_{\hat z}\, d\phi$ $\qquad + e_{\hat\varphi}\, d\theta$

axis e_z $\qquad$ axis e_ϕ

angle $d\phi$ $\qquad$ angle $d\theta$





$\nabla_Y X = 0$ constant vector field X if true for all Y

$\nabla_{j'} X^{i'} = e_{j'} X^{i'} + \Gamma^{i'}{}_{j'k'} X^{k'} = 0$ component version coord frames: $[e_i, e_j] = 0 = [e_{i'}, e_{j'}]$ — partial derivatives commute.

cartesian ($[e_i, e_j]$), spherical ($[e_{i'}, e_{j'}]$)

ON spherical not coordinate frame

$$e_{\hat i'} = (G_{i'i'})^{-1/2} e_{i'}$$ normalization factors

$$[e_{\hat i'}, e_{\hat j'}] = (G_{i'i'})^{-1/2} e_{i'}\big((G_{j'j'})^{-1/2}\big) e_{j'} - (G_{j'j'})^{-1/2} e_{j'}\big((G_{i'i'})^{-1/2}\big) e_{i'}$$

derivative terms along nonunit vectors

$$[e_{\hat r}, e_{\hat\theta}] = [e_r, r^{-1} e_\theta] = e_r(r^{-1})\, e_\theta = -\frac{1}{r^2} e_\theta = -\frac{1}{r} e_{\hat\theta}$$

$$[e_{\hat r}, e_{\hat\varphi}] = [e_r, r^{-1}(\sin\theta)^{-1} e_\varphi] = -\frac{1}{r^2\sin\theta} e_\varphi = -\frac{1}{r} e_{\hat\varphi}$$

$$[e_{\hat\theta}, e_{\hat\varphi}] = [r^{-1} e_\theta, r^{-1}(\sin\theta)^{-1} e_\varphi]$$

$$= r^{-1} e_\theta\big(r^{-1}(\sin\theta)^{-1}\big)\, e_\varphi = -r^{-2}(\sin\theta)^{-2}\cos\theta\, e_\varphi$$

$$- \big(r^{-1}(\sin\theta)^{-1} \underbrace{e_\varphi(r^{-1})}_{0}\big)\, e_\theta \qquad = -\frac{\cos\theta}{r\sin\theta}\,\frac{1}{r\sin\theta} e_\varphi$$

$$= -\frac{1}{r}\cot\theta\, e_{\hat\varphi}$$

easy to calculate.

later we learn why these are useful.

abstractly:

leave $\hat i'$ — re-express back

$$[e_{\hat i'}, e_{\hat j'}] = [B^m{}_i e_m, B^n{}_j e_n] = B^m{}_i (e_m B^n{}_j)\, e_n - B^n{}_j (e_n B^m{}_i)\, e_m$$

ON spherical

$$= (e_{\hat i'} B^n{}_j)(B^{-1\,p}{}_n e_{\hat p'}) - (e_{\hat j'} B^m{}_i)(B^{-1\,p}{}_m e_{\hat p'})$$

$$= \big(B^{-1\,p}{}_n\, e_{\hat i'} B^n{}_j - B^{-1\,p}{}_m\, e_{\hat j'} B^m{}_i\big]\, e_{\hat p'}$$

$$= \big([B^{-1} e_{\hat i'} B]^p{}_j - [B^{-1} e_{\hat j'} B]^p{}_i\big)\, e_{\hat p'}$$

$$= \big[\underbrace{[B^{-1}dB]^p{}_j(e_{\hat i'}) - [B^{-1}dB]^p{}_i(e_{\hat j'})}_{\text{antisymmetrized differential}}\big]\, e_{\hat p'}$$

antisymmetrized differential $\underbrace{B^{-1}dB}_{\text{antisymmetric}}$?

B orthogonal $\longrightarrow$