$\boxed{4.3}$ 1

$\boxed{\text{duality}}$

$\underset{\binom{1}{0}\text{ tensors}}{V, \{e_i\}} \leftrightarrow \underset{\binom{0}{1}\text{ tensors}}{V^*, \{\omega^i\}}$ both n-dim vectorspaces

$a_i x^i = a(x) = x(a) =$ scalar (ind. of basis)

natural pairing to produce scalars

interpretation $x : \{\alpha x\}$ = line (1-dim)

~~$a : a(x) = 0$: hyperplane (n-1 dim)~~

~~— same indexing at different levels~~

complementary subspaces (any multiple of a or x determines same spaces)

natural pairing extends to

$\Lambda\binom{p}{0} \leftrightarrow \Lambda\binom{0}{p}$ dual spaces

$S_{i_1 \dots i_p} T^{|i_1 \dots i_p|} =$ scalar $=$ " $S(T) = T(S)$ "

$\boxed{\text{star duality}}$

$$\underset{\text{p-vector}}{(S^{|i_1\dots i_p|} e_{i_1\dots i_p})} \wedge \underset{\text{q-vector}}{(T^{|i_{p+1}\dots i_n|} e_{i_{p+1}\dots i_n})}$$

$$= S^{|i_1\dots i_p|} T^{|i_{p+1}\dots i_n|} \underbrace{e_{i_1\dots i_p i_{p+1}\dots i_n}}_{\epsilon_{i_1\dots i_p i_{p+1}\dots i_n} e_{1\dots n}} = (S \wedge T)^{1\dots n} e_{1\dots n}$$

$$= \underbrace{(S^{|i_1\dots i_p|} \epsilon_{i_1\dots i_p i_{p+1}\dots i_n}) T^{|i_{p+1}\dots i_n|}}_{\text{scalar}} \underbrace{e_{1\dots n}}$$

$$\equiv (^*S)_{i_{p+1}\dots i_n}$$

n-p covector

BUT depends on basis

scalar $= [^*S](T)$

~~q-vector~~ covector p-vector

natural pairing to produce scalar.

~~— complementary indices~~

~~— shifts index position like ordinary dual~~

$\boxed{4.3}$ 2 Can repeat with opposite index level

$$(S_{|i_1 \cdots i_p|}\, \omega^{i_1 \cdots i_p}) \wedge (T_{|i_{p+1} \cdots i_n|}\, \omega^{i_{p+1} \cdots i_n}) = (S \wedge T)_{1 \cdots n}\, \omega^{1 \cdots n}$$

p-covector $\qquad$ (n-p) covector

~~etc~~

$$\cancel{S_{|i_1 \cdots i_p|}\, \epsilon^{i_1 \cdots i_p} \equiv (*S)^{i_{p+1} \cdots i_n}}$$ ~~(n-p) covector~~

$$\Lambda\binom{p}{0} \underset{*}{\overset{*}{\rightleftarrows}} \Lambda\binom{0}{n-p}$$

$* \circ * = (\text{sgn})$ Identity

determined by detailed formulas.

<u>easy in practice</u>

$$(B^{23} e_{23} + B^{31} e_{31} + B^{12} e_{12}) \wedge (E^1 e_1 + E^2 e_2 + E^3 e_3)$$

$$= B^{23} E^1 \underbrace{e_{231}}_{e_{123}} + B^{31} E^2 \underbrace{e_{312}}_{e_{123}} + B^{12} E^3 e_{123}$$

$$= (B^{23} E^1 + B^{31} E^2 + B^{12} E^3)\, e_{123}$$

$(*B)_1 \qquad (*B)_2 \qquad (*B)_3$

= linear function of E

$\therefore$ covector $*B$

$$(*B)_i = \epsilon_{ijk} B^{|jk|}$$

$$= (E^1 B^{23} + E^2 B^{31} + E^3 B^{12})\, e_{123}$$

linear function of B

2-covector

$(*E)_{23} \qquad (*E)_{31} \qquad (*E)_{12}$

$$(*E)_{ij} = \epsilon_{ijk} E^k$$

index formula → basis formula

$$T^{i_1 \dots i_p} \longrightarrow (*T)_{i_{p+1} \dots i_n} = T^{|j_1 \dots j_p|} \epsilon_{j_1 \dots j_p i_{p+1} \dots i_n}$$

$$*\left(T^{|i_1 \dots i_p|} e_{i_1 \dots i_p}\right) = (*T)_{i_{p+1} \dots i_n} \omega^{i_{p+1} \dots i_n}$$

$$= T^{|i_1 \dots i_p|} \underbrace{* e_{i_1 \dots i_p}} \qquad = T^{|j_1 \dots j_p|} \underbrace{\epsilon_{j_1 \dots j_p i_{p+1} \dots i_n} \omega^{|i_{p+1} \dots i_n|}}$$

$$* e_{i_1 \dots i_p} \overset{=}{=} \underbrace{\epsilon_{i_1 \dots i_p i_{p+1} \dots i_n}}_{\pm 1} \omega^{|i_{p+1} \dots i_n|}$$

complementary indices

$* : \binom{p}{0} \to \binom{0}{n-p}$

$* e_1 = \epsilon_{123} \omega^{23} = \omega^{23}$

$* e_2 = \epsilon_{231} \omega^{31} = \omega^{31}$

$* e_3 = \epsilon_{312} \omega^{12} = \omega^{12}$

$* e_{23} = \epsilon_{231} \omega^1 = \omega^1$

$* e_{31} = \epsilon_{312} \omega^2 = \omega^2$

$* e_{12} = \epsilon_{123} \omega^3 = \omega^3$

$* e_{123} = \epsilon_{123} \cdot 1 = 1$

$* 1 = \epsilon_{123} e_{123} = e_{123}$

similarly

$* \omega^1 = \epsilon^{123} e_{23} = e_{23}$

$* \omega^2$

$* \omega^3$

$* \omega^{23} = \epsilon^{231} e_1 = e_1$

$* \omega^{123} = \epsilon^{123} 1 = 1.$

$* 1 = \epsilon^{123} e_{123} = e_{123}$

but here $* : \binom{0}{p} \to \binom{n-p}{0}$

APART FROM INDEX LEVEL (up/down) (← need inner product)

only need 1 index for p-vectors / covectors in $\mathbb{R}^3$

"natural dual" $*$ ⟶ $(*)$ to distinguish from

inner product dual next

$\boxed{4.3}$ 4 $\underline{\mathbb{R}^4}$

$$(B^{[ijk]} e_{ijk}) \wedge (E^\ell e_\ell)$$

$$= \begin{pmatrix} B^{234} e_{234} \\ + B^{134} e_{134} \\ + B^{124} e_{124} \\ + B^{123} e_{123} \end{pmatrix} \wedge \begin{pmatrix} E^1 e_1 \\ + E^2 e_2 \\ + E^3 e_3 \\ + E^4 e_4 \end{pmatrix} = \begin{matrix} B^{234} E^1 e_{\widehat{2341}} = -e_{1234} \\ + B^{134} E^2 e_{\widehat{1342}} = +e_{1234} \\ + B^{124} E^3 e_{12\widehat{43}} = -e_{1234} \\ + B^{123} E^4 e_{1234} \end{matrix}$$

$$= (\underbrace{-B^{234}}_{(*B)_1} E^1 + \underbrace{B^{134}}_{(*B)_2} E_2 \underbrace{-B^{124}}_{(*B_3)} E^3 + \underbrace{B^{123}}_{(*B)_4} E^4)\, e_{1234}$$

$= B^{234} \epsilon_{\underbrace{2341}}$ ← from the formula

$= -B^{234}$ ✓ yes!

$$F = F^{[ij]} e_{ij} \longrightarrow *F_{mn} = F^{ij} \epsilon_{ijmn}$$

$$\begin{aligned} &= F^{41} e_{41} + F^{23} e_{23} \\ &\quad + F^{42} e_{42} + F^{31} e_{31} \\ &\quad + F^{43} e_{43} + F^{12} e_{12} \end{aligned}$$

$$*F_{14} = F^{23} \epsilon_{\widehat{23}14} = F^{23} = B^1$$

$$*F_{24} = F^{31} \epsilon_{\underbrace{31}24} = F^{31} = B^2$$

$$*F_{34} = F^{12} \epsilon_{1234} = F^{12} = B^3$$

$$\begin{aligned} \equiv\ & E^1 e_{41} + B^1 e_{23} \\ & + E^2 e_{42} + B^2 e_{31} \\ & + E^3 e_{43} + B^3 e_{12} \end{aligned}$$

$$*F_{23} = F^{41} \epsilon_{4123} = -F^{41} = -E^1$$

$(F^{ij}) \leftrightarrow (E^i, B^i)$

↑ top row

$(*F_{ij}) \longleftrightarrow (-B, -E^i)$

↑ top row

exchanges 2 3-vectors
with extra signs

$\boxed{4.3}$ 5

problem: $\epsilon_{i_1 \ldots i_n} \to (\det A^{-1})\, \epsilon_{i_1 \ldots i_n}$ basis change.

all stardual relations ~~shift~~ scale.

because $e_{1\ldots n} = e_1 \wedge \ldots \wedge e_n$ depends on basis up to a scalar.

~~If we can fix 1 particular n-vector & define stardual in terms of its coefficients will be basis independent.~~

fix inner product G (dot product on $\mathbb{R}^n$)

pick an ordering of an orthonormal basis

$\{e_i\} \to e_1 \wedge \ldots \wedge e_n = e_{1\ldots n}$ fixes basis n-vector "unit volume" n-vector.

$\downarrow$

$e_j O^j{}_i$ (pseudo) orthonormal basis change

~~$\det(\underline{O}) = \pm 1$~~

$\downarrow$

$e_{1\ldots n} \to \pm e_{1 \ldots n} \leftarrow$ choose + sign for ~~oriented basis~~ − sign for opposite orientation.

$e_{1\ldots n}$ picks out unit n-dim rectangular box volume, fixes scale for any other bases.

well defined up to sign — need to choose one

extra structure = "orientation"

$\boxed{4.3}$ G

non orthonormal frames

$$e_{j'} = \underset{\sim}{e_j} A^{-1 j}{}_i = e_j B^j{}_i$$

not ON

$$e_{1' \ldots n'} = (\det \underline{B})\, \underbrace{e_{1 \ldots n}}_{\text{unit volume}} \longrightarrow e_{1 \ldots n} = \frac{1}{\det \underline{B}}\, e_{1' \ldots n'}$$

corrects for parallelopiped volume of new basis

But $\underline{G}' = \underline{B}^T \underline{G}\, \underline{B}$

$$\det \underline{G}' = \det(\underline{B}^T \underline{G}\, \underline{B}) = \underbrace{\det \underline{B}^T}_{\det \underline{B}} \det \underline{G} \det \underline{B}$$

$$= (\det \underline{B})^2 \det \underline{G}$$

$$\sqrt{|\det \underline{G}'|} = |\det \underline{B}|\, \underbrace{\sqrt{|\det \underline{G}|}}_{=1 \text{ ON basis}} \longrightarrow$$

$$\eta^\# = e_{1 \ldots n} = \frac{1}{\sqrt{\det \underline{G}'}}\, e_{1' \ldots n'}$$

up components

$$\boxed{\eta^{i_1' \ldots i_n'} = \frac{1}{\sqrt{\det \underline{G}'}}\, \epsilon^{i_1 \ldots i_n}}$$

oriented n-vector

sets volume scale.

not basis dependent in oriented frames

$$e_{1' \ldots n'} = \alpha\, e_{1 \ldots n}$$

$\alpha > 0$

$\boxed{4.3}$ 7

$\eta_{i_1' \dots i_n'} = \sqrt{|\det \underline{G}'|}\, \epsilon_{i_1 \dots i_n}$

$\uparrow$

η unit n-covector.

use $\eta, \eta^{\#}$ instead of ϵ to define

basis independent dual

can raise/lower indices.

inner product star dual

$\binom{p}{0} \rightarrow \binom{n-p}{0}$ shift indices so dual has indices at same level.

$T^{i_1 \dots i_p} \rightarrow T^{|j_1 \dots j_p|} \eta_{j_1 \dots j_p i_{p+1} \dots i_n}$ $\uparrow$ raise indices

$T^{|j_1 \dots j_p|} \eta_{j_1 \dots j_p}{}^{i_{p+1} \dots i_n} \equiv {*T}^{i_{p+1} \dots i_n}$

$T_{i_1 \dots i_p} \rightarrow T_{|j_1 \dots j_p|} \eta^{j_1 \dots j_p i_{p+1} \dots i_n}$ $\downarrow$ lower indices

$T_{|j_1 \dots j_p|} \eta^{j_1 \dots j_p}{}_{i_{p+1} \dots i_n} \equiv {*T}_{i_{p+1} \dots i_n}$