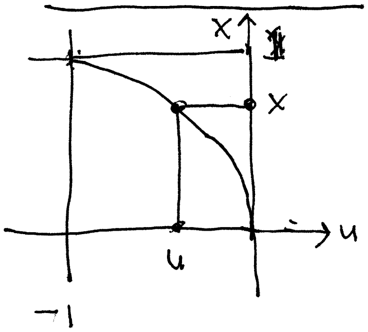
$\boxed{11.1}\boxed{4}$

1

DIFFERENTIAL FORMS

change of variable



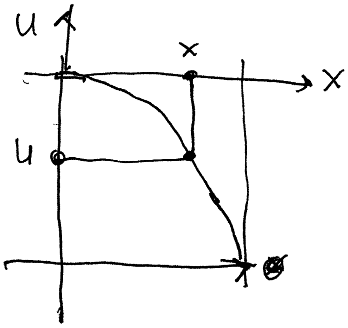
$u = -x^2$

$x = (-u)^{1/2} = g(u)$

$\frac{dx}{du} \neq 0$, sgn fixed

> 0 increasing

< 0 decreasing



$$\overbrace{\int_{x_1}^{x_2} f(x)\,dx = \int_{u_1}^{u_2}}^{} f(g(u))\,\underbrace{\frac{dx}{du}(u)}_{=\pm\left|\frac{dx(u)}{du}\right|}\,du = \int_{\min(u_1,u_2)}^{\max(u_1,u_2)} f(g(u))\,\underbrace{\left|\frac{dx(u)}{du}\right|\,du}_{\text{"}ds\text{"} > 0}$$

$x_1 \le x_2$

$x_i = g(u_i)$

now ordered

$\begin{pmatrix} u \text{ increasing} \\ du > 0 \end{pmatrix}$

$dx = ds$ on $\mathbb{R}$

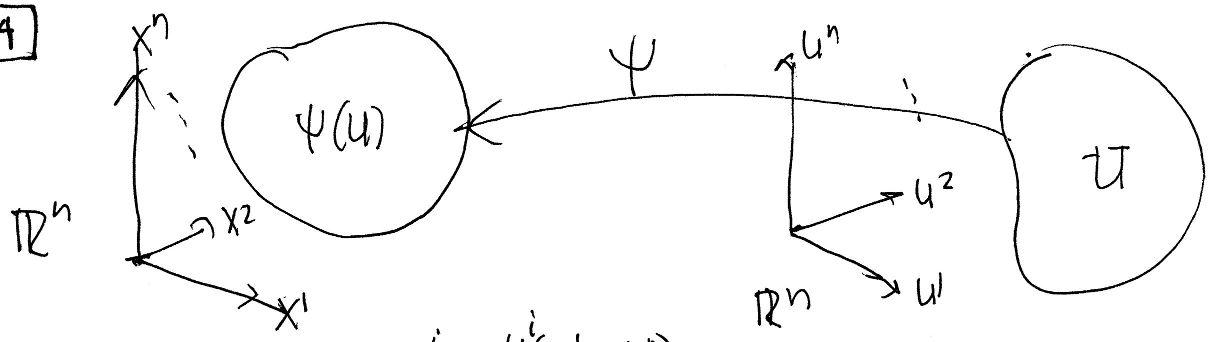
1-d Jacobian $\frac{dx}{du}$ is correction factor to produce arclength

example: $\int_0^1 e^{-x^2} x\,dx > 0$

$$= \int_0^{-1} e^u \left(-\tfrac{1}{2}\right) du = \int_{-1}^{0} e^u \left(\tfrac{1}{2}\right) du = \tfrac{1}{2} e^u \Big|_{u=-1}^{u=0} = \tfrac{1}{2}(1 - e^{-1}) > 0$$

switch limits & sign

$\boxed{11.1}\boxed{-.4}$
2



$$x^i = \psi^i(u^1,\dots,u^n) = \psi^i(u)$$

$$\int\cdots\int_{\psi(U)} f(x^1,\dots,x^n)\,\underbrace{dx^1 dx^2\cdots dx^n}_{dV_x}$$

$$= \int\cdots\int_{U} f(\psi(u^1,\dots,u^n))\,\underbrace{\underbrace{\left|\det\left(\frac{\partial\psi^i}{\partial u^j}\right)\right|}_{\text{correction factor}}\,\underbrace{du^1 du^2\cdots du^n}_{dV_u}}_{dV_x}$$

If $f \geq 0$ everywhere in $\psi(U)$
integral ≥ 0

$$(x,y,z) \to (\rho,\varphi,z) : \overset{dx\,dy\,dz \downarrow}{\rho\, d\rho\, d\phi\, dz}$$

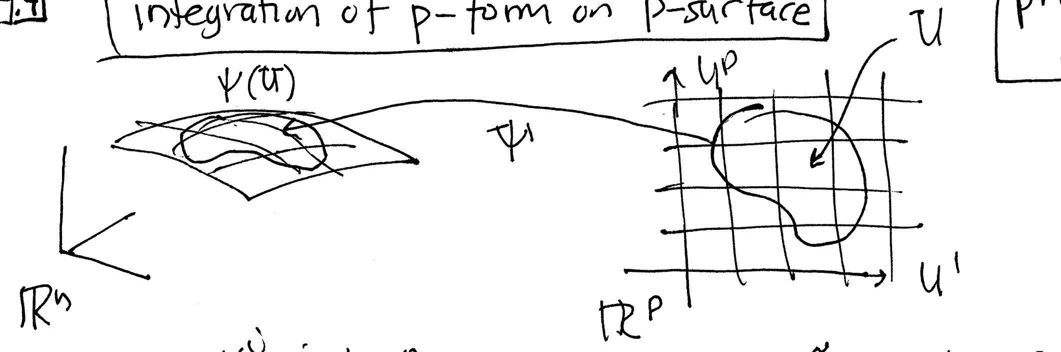
$$\to (r,\theta,\varphi) : r^2\sin\theta\, dr\, d\theta\, d\phi$$

$\boxed{\text{II.H.4}}$

3

$\boxed{\text{integration of p-form on p-surface}}$

$\boxed{\text{preliminary stuff}}$



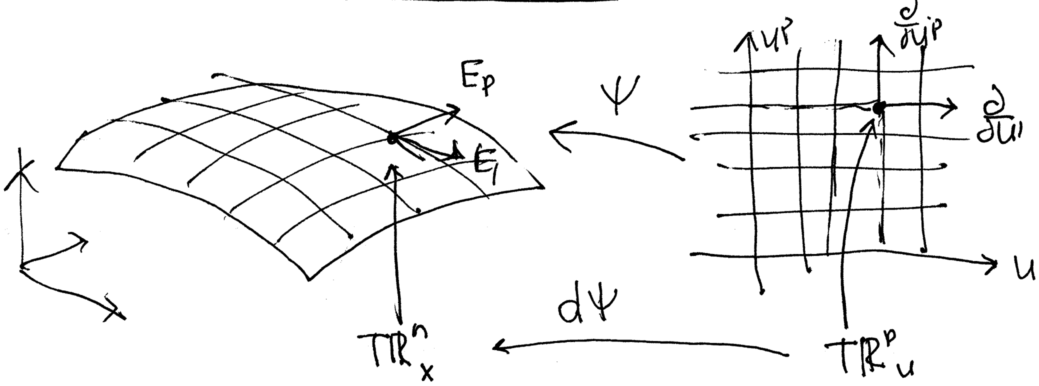
$x^i, i=1,\dots,n$ $\qquad\qquad$ $u^\alpha, \alpha=1,\dots,p$

$\underline{x^i = \Psi^i(u)}$:

p-surface coord grid $\longleftarrow$ coordinate grid

$$E_\alpha^i(X(u)) = \frac{\partial x^i(u)}{\partial u^\alpha}\frac{\partial}{\partial x^i}\Big|_{x=x(u)} \xleftarrow{\text{chain rule}} \frac{\partial}{\partial u^\alpha} \quad \text{coord frame on } \mathbb{R}^p$$

surface coord frame $\qquad \simeq \left.\dfrac{\partial}{\partial u^\alpha}\right|_{x(u)}$ coord frame on surface



$$dx^i(u) = d\Psi^i(u) = \frac{\partial\psi^i}{\partial u^\beta}du^\beta \quad \text{differentials of coord functions}$$

$$d\psi^i(u)\left(\frac{\partial}{\partial u^\alpha}\right) = \frac{\partial\psi^i(u)}{\partial u^\beta}\underbrace{du^\beta\left(\frac{\partial}{\partial u^\alpha}\right)}_{\delta^\beta{}_\alpha} = \frac{\partial\Psi^i(u)}{\partial u^\alpha} \quad \text{components of vector for each } \alpha$$

$$d\Psi(u)\left(\frac{\partial}{\partial u^\alpha}\Big|_u\right) \equiv \frac{\partial\psi^i(u)}{\partial u^\alpha}\frac{\partial}{\partial x^i}\Big|_{\Psi(u)} \equiv E_\alpha\cancel{(\Psi(u))}\Big|_{\Psi(u)}$$

$\in T\mathbb{R}^p_u$ $\qquad\qquad$ $\in T\mathbb{R}^n_{\Psi(u)}$

$\Psi: \mathbb{R}^p \longrightarrow \mathbb{R}^n$

$d\Psi: T\mathbb{R}^p_u \to T\mathbb{R}^n_{\Psi(u)}$ $\qquad$ $\boxed{\text{push forward}}$ of tangent vectors from $T\mathbb{R}^p$ to $T\mathbb{R}^n$

same directions $\qquad$ if Ψ 1-1 map → push forward vector fields

$$X(u) = X^\alpha(u)\frac{\partial}{\partial u^\alpha} \longrightarrow d\Psi(X)(u) = X^\alpha(u)\, d\Psi(u)\left(\frac{\partial}{\partial u^\alpha}\right) = X^\alpha(u)\, E_\alpha(u)$$

$$X(u)f = X^\alpha(u)\frac{\partial f(\Psi(u))}{\partial u^\alpha} \leftarrow \text{first express } f(x) = f(x(u)), \text{ then diff.}$$

$\{E_\alpha(u)\} \overset{``\;"}{=} \{\frac{\partial}{\partial u^\alpha}\}$ coord frame on p-surface

$$E_1(u)\wedge\ldots\wedge E_p(u) \longleftrightarrow [E_1(u)\wedge\ldots\wedge E_p(u)]^{i_1\ldots i_p} = p!\; E_1(u)^{[i_1}\ldots E_p(u)^{i_p]}$$

$\neq 0$
linearly ind vectors
at each point

↑ removes p! in antisymmetric part

$\boxed{\text{Any } \binom{m}{0}\text{-tensor can be pushed forward}}$

example: $T = T^{\alpha\beta}\frac{\partial}{\partial u^\alpha}\otimes\frac{\partial}{\partial u^\beta} \xrightarrow{d\psi} T^{\alpha\beta}\circ\psi\;\underbrace{d\psi(\frac{\partial}{\partial u^\alpha})\otimes d\psi(\frac{\partial}{\partial u^\beta})}_{E_\alpha(u)\otimes E_\beta(u)}$

$= "\,T^{\alpha\beta}\frac{\partial}{\partial u^\alpha}\otimes\frac{\partial}{\partial u^\beta}\,"$ in coords u^α on p-surface

curve on $\mathbb{R}^p$: $\quad C(t) \leftrightarrow C^\alpha(t) = u^\alpha(C(t))$

$$C'(t) = C^{\alpha\,\prime}(t)\frac{\partial}{\partial u^\alpha}$$

$$d\psi(u)(C'(t)) = C^{\alpha\,\prime}(t)\;\underbrace{d\psi(u)(\frac{\partial}{\partial u^\alpha})}_{``\frac{\partial}{\partial u^\alpha}"\text{ on p-surface}}$$

$$d\psi(u)(C'(t))\,f$$

$$= C^{\alpha\,\prime}(t)\frac{\partial}{\partial u^\alpha}f(\psi(c(t)))$$

express f in terms of u

↑ diff on $\mathbb{R}^p$ $\quad f\circ\psi$ now function on $\mathbb{R}^p$

$$\begin{tikzcd} u \arrow[r] & \psi(u) \\ \in\mathbb{R}^p & \in\mathbb{R}^n \\ f\circ\psi \text{ on } \mathbb{R}^p & f \text{ on } \mathbb{R}^n \arrow[l] \end{tikzcd}$$

$\boxed{\text{pullback of } f}$

functions go opposite direction under composition with map.

$\psi^* f \equiv f\circ\psi$

$$\psi^* f = f \circ \psi$$

$$d(\psi^* f) = \frac{\partial f}{\partial x^i} \cancel{(}\psi \, \frac{\partial x^i}{\partial u^\alpha} \, du^\alpha \qquad \text{(chain rule again)}$$

$$df = \frac{\partial f}{\partial x^i} dx^i \longrightarrow d(\psi^* f) = \left(\frac{\partial f}{\partial x^i} \circ \psi \, \frac{\partial x^i}{\partial u^\alpha} \right) du^\alpha$$

$$\frac{\partial f}{\partial x^i} \longrightarrow \frac{\partial x^i}{\partial u^\alpha} \frac{\partial f}{\partial x^i} \qquad \text{covector transformation}$$

$$\equiv \psi^* df \qquad \text{pullback of } df \text{ to } \mathbb{R}^p.$$

$$\begin{tikzcd}
\mathbb{R}^p \arrow[r] & \mathbb{R}^n \\
\text{functions} & \text{functions} \arrow[l, "\psi^*"] \\
\text{1 forms} & \text{1-forms} \arrow[l]
\end{tikzcd}$$

$$\psi^* x^i = x^i \circ \psi = \psi^i$$

$$\psi^* dx^i = \frac{\partial \psi^i}{\partial u^\alpha} du^\alpha = \text{"}dx^i(u)\text{"}$$

$$\underbrace{d\psi\left(\frac{\partial}{\partial u^\alpha}\right)}_{\text{push forward of vector acting on } f} f = \frac{\partial}{\partial u^\alpha} \underbrace{(f \circ \psi)}_{\text{pull back of } f \text{ acted on by original vector}}$$

all just re-interpretations of chain rule

extend pullback to $\begin{pmatrix} 0 \\ m \end{pmatrix}$-tensors: just plug in $x = x(u)$ everywhere.

example: $g = g_{ij}(x)\, dx^i \otimes dx^j$.

$$\psi^* g = g_{ij}(x(u))\, dx^i(u) \otimes dx^j(u) = \Big(\underbrace{g_{ij}(x(u)) \frac{\partial x^i}{\partial u^\alpha}(u) \frac{\partial x^j}{\partial u^\beta}(u)}_{g_{\alpha\beta}(u)} \Big) du^\alpha \otimes du^\beta$$

$$g_{\alpha\beta}(u) = g(E_\alpha(u), E_\beta(u)) = g\left(\frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta}\right)$$

pullback of metric on $\mathbb{R}^n$ to metric on $\mathbb{R}^p$

$= \boxed{\text{induced metric on } p\text{-surface}}$ expressed in coords u^α

p-form on $\mathbb{R}^n$:

$$T = \frac{1}{p!} T_{i_1\dots i_p}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}$$

$$= T_{\underbrace{|i_1\dots i_p|}_{\text{ordered p-tuples of indices to avoid overcounting}}}\, dx^{i_1}\wedge\dots\wedge dx^{i_p}$$

just plug in everywhere
$x^i \overset{''}{=} x^i(u) \overset{''}{=} \psi^i(u)$

$$\psi^* T = \frac{1}{p!} T_{i_1\dots i_p}\circ\psi \quad d\psi^{i_1}\wedge\dots\wedge d\psi^{i_p}$$

$$\frac{\partial\psi^{i_1}}{\partial u^{\alpha_1}} du^{\alpha_1}\wedge\dots\wedge\frac{\partial\psi^{i_p}}{\partial u^{\alpha_p}} du^{\alpha_p}$$

$$\frac{\partial\psi^{i_1}}{\partial u^{[\alpha_1}}\cdots\frac{\partial\psi^{i_p}}{\partial u^{\alpha_p]}} \underbrace{du^{\alpha_1}\wedge\dots\wedge du^{\alpha_p}}_{du^1\wedge\dots\wedge du^p\, \epsilon^{\alpha_1\dots\alpha_p}} \qquad \text{p-form on } \mathbb{R}^p$$

$$= \frac{1}{p!} \underbrace{T_{i_1\dots i_p}}_{\text{antisym}}\circ\psi \underbrace{\frac{\partial\psi^{i_1}}{\partial u^{\alpha_1}}\cdots\frac{\partial\psi^{i_p}}{\partial u^{\alpha_p}}\epsilon^{\alpha_1\dots\alpha_p}}_{p!\frac{\partial\psi^{i_1}}{\partial u^1}\cdots\frac{\partial\psi^{i_p}}{\partial u^n}} \quad du^1\wedge\dots\wedge du^p$$

$$= T_{i_1\dots i_p}\circ\psi \underbrace{\frac{\partial\psi^{[i_1}}{\partial u^1}\cdots\frac{\partial\psi^{i_p]}}{\partial u^p}}_{\underbrace{E_1^{[i_1}\cdots E_p^{i_p]}}_{\frac{1}{p!}(E_1\wedge\dots\wedge E_p)^{i_1\dots i_p}}} \quad du^1\wedge\dots\wedge du^p$$

$$(E_1\wedge\dots\wedge E_p)^{i_1\dots i_p} \equiv p!\, E_1^{[i_1}\cdots E_p^{i_p]}$$

$$= \underbrace{\frac{1}{p!} T_{i_1\dots i_p}\circ\psi\, (E_1\wedge\dots\wedge E_p)^{i_1\dots i_p}} \quad du^1\wedge\dots\wedge du^p$$

$$= \underbrace{T(E_1,\dots,E_p)}_{\text{value of p-form on coord frame on p-surface}} = T\left(\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^p}\right) \quad \text{expressed in coords on p-surface}$$

$\boxed{11.1}$ $\boxed{7.9}$ Integration of p-form on p-surface (oriented by Ψ)

7

First p-form on $\mathbb{R}^p$ →

$$\int_U \underbrace{F(u)}_{\text{if } \geq 0 \text{ the integral } \geq 0} du^1\wedge\dots\wedge du^p \equiv \underbrace{\int\dots\int_U}_{\text{all limits}} F(u)\, du^1\dots du^p \quad \left(\begin{matrix}\text{Iterated}\\ \text{integral}\end{matrix}\right)$$

all limits $u^i = \underbrace{u^i_{\text{initial}} \cdots u^i_{\text{final}}}_{\text{ordered limits}}$

if ≥ 0 the integral ≥ 0 → ordered limits

now:

$$\int_{\Psi(U)} T \equiv \int_U \Psi^* T = \int\dots\int_U T_{|i_1\dots i_p|}\circ\Psi \,(E_1\wedge\dots\wedge E_p)^{i_1\dots i_p} du^1\dots du^p$$

to avoid overcounting.

Example
$x^1 = r_0 \sin u^1 \cos u^2$
$x^2 = r_0 \sin u^1 \sin u^2$
$x^3 = r_0 \cos u^1$

↔

$x = r_0 \sin\phi \cos\theta$
$y = r_0 \sin\phi \sin\theta$
$z = r_0 \cos\phi$

$S^2(r_0)$ sphere of radius r_0 on $\mathbb{R}^3$.

$$\iint_{S^2(r_0)} x_1\, dx^2\wedge dx^3 + x_2\, dx^3\wedge dx^1 + x_3\, dx^1\wedge dx^2 = \iint_{S^2(r_0)} x\, dy\wedge dz + y\, dz\wedge dx + z\, dx\wedge dy$$

$$= \iint_{S^2(r_0)} (r_0 \sin\phi\cos\theta)\, d(r_0\sin\phi\sin\theta)\wedge d(r_0\cos\phi)$$
$$+ (r_0\sin\phi\sin\theta)\, d(r_0\cos\phi)\wedge d(r_0\sin\phi\cos\theta)$$
$$+ (r_0\cos\phi)\, d(r_0\sin\phi\cos\theta)\wedge d(r_0\sin\phi\sin\theta)$$

$$= \iint_{S^2(r_0)} r_0^3 \Big[\sin\phi\cos\theta\,(\cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta)\wedge(-\sin\phi\, d\phi)$$
$$+ \sin\phi\sin\theta\,(-\sin\phi\, d\phi)\wedge(\cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta)$$
$$+ \cos\phi\,(\cos\phi\cos\theta\, d\phi - \sin\phi\sin\theta\, d\theta)\wedge(\cos\phi\sin\theta\, d\phi + \sin\phi\cos\theta\, d\theta)\Big]$$

$$= \iint_{S^2(r_0)} r_0^3 \Big[\sin\phi\cos\theta\,(-\sin^2\phi\cos\theta)(d\theta\wedge d\phi)$$
$$+ \sin\phi\sin\theta\,(\sin^2\phi\sin\theta)(d\phi\wedge d\theta)$$
$$+ \cos\phi\,(\cos^2\phi\sin\phi\cos^2\theta)\, d\phi\wedge d\theta + \cos\phi\,(-\sin\phi\cos\phi\sin^2\theta)(d\theta\wedge d\phi)$$

$$= \iint_{S^2(r_0)} r_0^3 \sin\phi \Big[\underbrace{\underbrace{-\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta}_{-\sin^2\phi} \; \underbrace{-\cos^2\phi\cos^2\theta - \cos^2\phi\sin^2\theta}_{-\cos^2\phi}}_{-1}\Big]\, d\theta\wedge d\phi$$

$$= \iint_{S^2(r_0)} r_0^3 \sin\phi\, d\phi\wedge d\theta \equiv \int_0^{2\pi}\int_0^{\pi} r_0\, \underbrace{(r_0^2\sin\phi)\, d\phi\, d\theta}_{dS} = r_0\left(\frac{4\pi r_0^3}{3}\right).$$

$$= \iint |\vec{x}|\, dS = \iint \vec{x}\cdot d\vec{S} \leftrightarrow \text{"}\, d\vec{S} = \langle dy\wedge dz, dz\wedge dx, dx\wedge dy\rangle \,\text{"}$$

$\boxed{11.1}\boxed{-A}$

8

$$\iint_{S^2} B_1\, dy\wedge dz + B_2\, dz\wedge dx + B_3\, dx\wedge dy$$

no metric

$$= \iint_{S^2} \underbrace{\vec{B}\cdot\hat{n}}_{B_\perp}\ \underbrace{dS}_{\sqrt{\det(g^{(2)})} \,=\, r_0^2\sin\phi}$$

$$g^{(2)} = r_0^2(d\phi\otimes d\phi + \sin^2\phi\, d\theta\otimes d\theta)$$

since $|\vec{X}| = \vec{X}\cdot\hat{X} = \vec{X}\cdot\hat{n}$ unit outward normal component.

can re-interpret metricless integration process in terms of metric quantities for interpretation

$$\iint_\Sigma B = \iint \underbrace{({}^*B)^\#}_{\text{vector field}}\cdot\hat{n}\, dS = \iint ({}^*B^\#)_\perp\, dS$$

B: 2-form
↓
dual is 1-form
↓
raise index → vector field

outward normal component along $\hat{n}$

need choice of normal direction

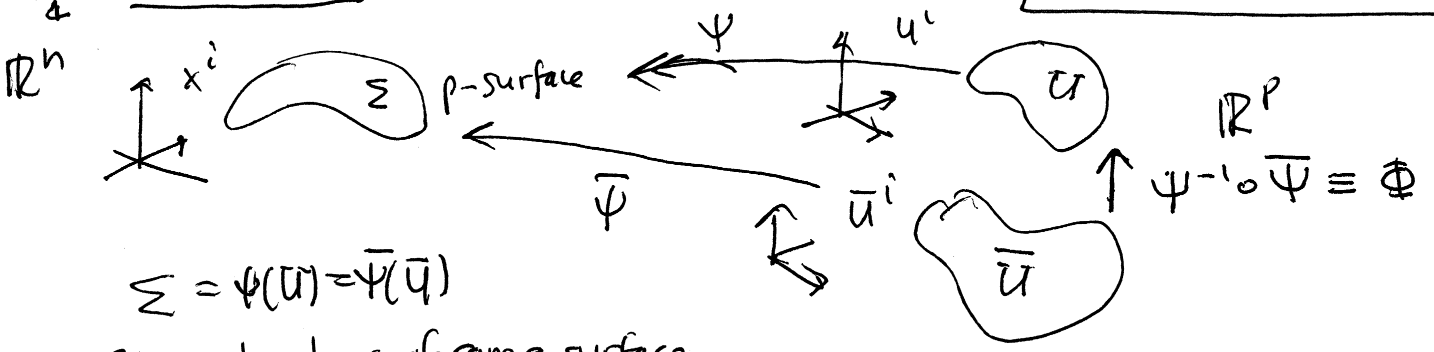
parametrized surface automatically defines orientation by $E_1\wedge..\wedge E_p$ has "positive" orientation

more next time

$\boxed{11.5}$ ~~summary only (read details)~~

4

$\boxed{\text{Integration of p-form on p-surface needs orientation}}$



$\Sigma = \psi(\bar U) = \bar\Psi(\bar U)$

2 parametrizations of same surface

$$\int_{\psi(U)} T = \int\cdots\int_{U} \frac{1}{p!} T_{i_1\ldots i_p}\circ\psi(u)\, \underbrace{[\overset{\overset{\partial x^i}{\partial u^1}}{E_1}(u)\wedge\ldots\wedge E_p(u)]^{i_1\ldots i_p}}_{\text{volume "det" correction factor}}\, du^1 du^2\ldots du^p$$

→ integrand $= T(E_1,\ldots,E_p)$ $\underline{\underline{\text{scalar}}}$

$$\int_{\bar\Psi(u)} T = \int\cdots\int_{\bar U} \frac{1}{p!} T_{i_1\ldots i_p}\circ\bar\Psi(\bar u)\, \underbrace{[\overset{\overset{\partial x^i}{\partial \bar u^1}}{\bar E_1}(\bar u)\wedge\ldots\wedge \bar E_p(\bar u)]^{i_1\ldots i_p}}_{\text{ditto}}\, d\bar u^1 d\bar u^2\ldots d\bar u^p$$

↳ integrand $= T(\bar E_1,\ldots,\bar E_p)$ $\underline{\underline{\text{scalar}}}$

$\psi(u) = \bar\Psi(\bar u)$

↳ $u = \psi^{-1}\circ\bar\Psi(\bar u) \equiv \Phi(\bar u)$ ↳ $\bar\Psi^i(\bar u) = \Psi^i(\Phi(\bar u))$ CHAIN RULE

$$\underbrace{\bar E^i{}_\alpha(\bar u) = \frac{\partial\bar\Psi^i(\bar u)}{\partial\bar u^\alpha} = \frac{\partial}{\partial\bar u^\alpha}\psi^i(\phi(\bar u)) = \frac{\partial\psi^i}{\partial\bar u^\beta}(\phi(\bar u))\,\frac{\partial\Phi^\beta}{\partial\bar u^\alpha}(\bar u) = E^i{}_\beta(\phi(\bar u))\,\frac{\partial\phi^\beta}{\partial\bar u^\alpha}(\bar u)}_{\text{change of coords on } \mathbb{R}^p \text{ : transformation}}$$

$$\bar E_1(\bar u)\wedge\ldots\wedge\bar E_p(\bar u) = E_{\alpha_1}(\phi(\bar u))\frac{\partial\phi^{\alpha_1}}{\partial\bar u^1}(\bar u)\wedge\ldots\wedge E_{\alpha_p}(\phi(\bar u))\frac{\partial\phi^{\alpha_p}}{\partial\bar u^p}(\bar u)$$

$$= \underbrace{(E_{\alpha_1}\wedge\ldots\wedge E_{\alpha_p})}_{\epsilon_{\alpha_1\ldots\alpha_p}\ E_1\wedge\ldots\wedge E_p}(\phi(\bar u))\ \underbrace{\left(\frac{\partial\phi^{\alpha_1}}{\partial\bar u^1}\cdots\frac{\partial\phi^{\alpha_p}}{\partial\bar u^p}\right)}_{\det\left(\frac{\partial\phi^\alpha}{\partial\bar u^\beta}\right)}(\bar u)$$

Jacobian determinant

drop function dependence

$$= \det\left(\frac{\partial\phi^\alpha}{\partial\bar u^\beta}\right) E_1\wedge\ldots\wedge E_p$$

differ only by det of p×p matrix → $\bar E_\alpha = \frac{\partial u^\beta}{\partial\bar u^\alpha} E_\beta$

$\boxed{11.5}$

2

$$\int_{\Psi(\bar{u})} T = \int\cdots\int_{\bar{u}} \frac{1}{p!} T_{i_1\ldots i_p} \circ \bar{\Psi}(\bar{u})\, E_1(u(\bar{u}))\wedge\ldots\wedge E_p(u(\bar{u}))^{i_1\ldots i_p}\, \underbrace{\det\left(\frac{\partial \phi^\alpha}{\partial \bar{u}^\beta}\right) d\bar{u}^1 \ldots d\bar{u}^p}_{}$$

$$= \int\cdots\int_{u} \frac{1}{p!} T_{i_1\ldots i_p}(u)\, E_1(u)\wedge\ldots\wedge E_p(u)^{i_1\ldots i_p} \qquad du^1\ldots du^p$$

change of variable of integral of function on $\mathbb{R}^p$ $\qquad du^1\ldots du^p \to \left|\det\left(\frac{\partial u^\alpha}{\partial \bar{u}^\beta}\right)\right| d\bar{u}^1\ldots d\bar{u}^p$

will agree if $\det\left(\frac{\partial u^\alpha}{\partial \bar{u}^\beta}\right) > 0$

changes sign if $\det\left(\frac{\partial u^\alpha}{\partial \bar{u}^\beta}\right) < 0$ } to be sign independent must pick "orientation" for surface

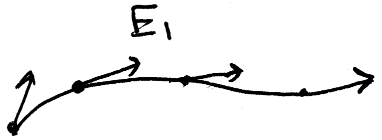
choice of any $E_1\wedge\ldots\wedge E_p$ associated with some parametrization gives "positive orientation" — just an ordering of the vectors

if parametrization $\bar{E}_1\wedge\ldots\wedge\bar{E}_p = k\, E_1\wedge\ldots\wedge E_p$

$k > 0$ positively oriented, keep sign

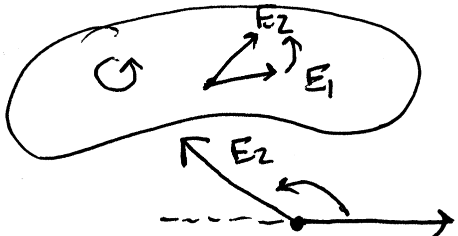
$k < 0$ negatively oriented, change sign of integral

$p=1$

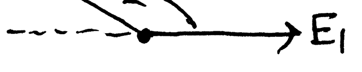


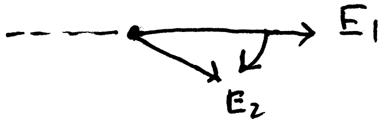
directed curve

$p=2$



circulation sense on surface "screw sense"

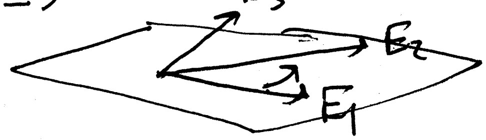




rotate E1 towards E2 without crossing − E1

curl fingers

$p=3$



righthand rule: $\overbrace{\text{rotate } E_1 \text{ towards } E_2}^{\text{curl fingers}}$ as in case $p=2$, thumb picks out side of plane of E_1, E_2 E_3 lies

$\boxed{11.6}$

1

$\int_\Sigma F$ needs no metric, but if a metric is present we can translate it into an expression involving that metric to interpret the value

Σ: oriented p surface

F: p-form

$\underbrace{\int_\Sigma F}$ = real #

$$g = g_{ij}\, dx^i \otimes dx^j$$

---

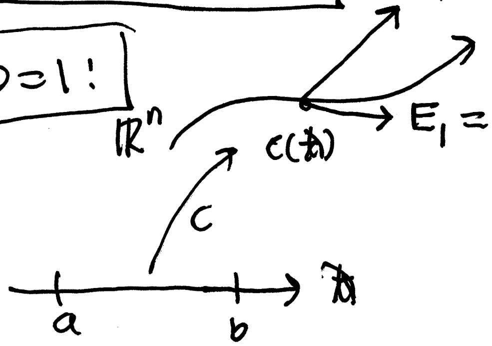
what does it mean?

$\vec{F} = F^\#$

$F = F_j\, dx^j$ 1-form

$\hookrightarrow F^i = g^{ij} F_j$ vector field

$\boxed{p=1!}$



$$E_1 = c'(\lambda) = \text{“}\frac{dx^i}{d\lambda}\frac{\partial}{\partial x^i}\text{”}$$

$$= \epsilon \underbrace{\left|\frac{dx}{d\lambda}\right|}_{\text{length}} \underbrace{\hat{T}^i(\lambda)}_{\text{unit tangent.}}\frac{\partial}{\partial x^i}$$

$\hat{T}\cdot\hat{T} = \epsilon = \pm 1$

$\hat{T}\cdot\frac{d\vec{x}}{dt}$

$= \epsilon\left|\frac{d\vec{x}}{d\lambda}\right| \hat{T}\cdot\hat{T}$

$= \left|\frac{d\vec{x}}{d\lambda}\right| > 0$

$$\int_C F = \int_C F_i\, dx^i = \int_a^b c^*(F) = \int_a^b F_i(c(\lambda))\underbrace{\frac{dx^i}{d\lambda}(\lambda)\, d\lambda}_{\epsilon\left|\frac{dx}{d\lambda}\right|\hat{T}^i d\lambda}$$

plug in $x^i = x^i(u)$

$$= \int_a^b \underbrace{\epsilon F_i(c(\lambda))\hat{T}^i(\lambda)}_{F^{\parallel}(c(\lambda))} \underbrace{\left|\frac{dx(\lambda)}{d\lambda}\right| d\lambda}_{ds(\lambda)}$$

$F^{\parallel}(c(\lambda))$ tangential component along $\hat{T}$

$$\frac{ds(\lambda)}{d\lambda} = \left|\frac{dx}{d\lambda}(\lambda)\right|$$

“speed” = length tangent vector = “velocity”

$\boxed{= \text{integral of tangential component of “}\vec{F}\text{”} = F^\# \text{ with respect to differential of arclength}}$

usual line integral from multivariable calculus

$$\boxed{F(c'(\lambda))\, d\lambda = \epsilon \vec{F}\cdot c'(\lambda)\, ds}$$

no metric — metric

$\boxed{11.6}$

2

$p=2$ in $\mathbb{R}^3$ with flat metric g but any signature — flat coords $\{x^i\}$

surface with unit normal $\hat n^i$: $\hat n^i \hat n_i = \epsilon = \pm 1$

ϵ_{kij}

2-form $F = \frac{1}{2} F_{ij}\, dx^i \wedge dx^j = \underbrace{\frac{1}{2} (*B)_{ij}\, dx^i \wedge dx^j = \frac{1}{2} B^k \eta_{kij}}_{\text{definition}}\, dx^i \wedge dx^j$

= dual of 1-form B

$\updownarrow$

vector field $\vec B = B^\sharp$

$\eta(\vec B, \,, \,)$ evaluated on $\vec B$ in first slot

$\boxed{\text{see Part 1}}$

ordered

$= F_{|ij|}\, dx^i \wedge dx^j$ $\quad \epsilon_{213} = -1$

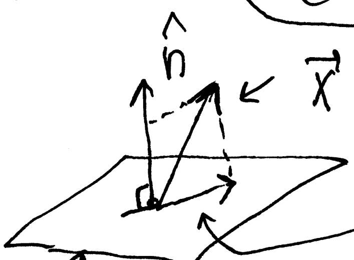
$\epsilon_{123} = 1$ $\quad \epsilon_{312} = 1$

$= B^1 dx^2 \wedge dx^3 - B^2 dx^1 \wedge dx^3 + B^3 dx^1 \wedge dx^2$

$= B^1 dx^2 \wedge dx^3 + B^2 dx^3 \wedge dx^1 + B^3 dx^1 \wedge dx^2$ ← cyclic sum "nicer"

(but alternating sign generalizes to higher dimension)

Detour:



$\vec X = X^\perp \hat n + P(\hat n) \vec X$

orthogonal decomposition of tangent space for any X

tangent plane to surface

$\hat n \cdot \hat n = \epsilon = \pm 1$

$$P(\hat n)^i{}_j \equiv - \frac{\hat n^i \hat n_j}{\hat n \cdot \hat n} + \delta^i{}_j \rightarrow (\delta^i{}_j = \epsilon \hat n^i \hat n_j + P^i{}_j)$$

$$P(\hat n)^i{}_j B^j = - \hat n^i \overset{\equiv B^\perp}{\frac{\hat n_j B^j}{\hat n \cdot \hat n}} + \delta^i{}_j B^j = B^i - \underbrace{B^\perp \hat n^i}_{\text{subtract away normal component}}$$

$B^i = B^\perp \hat n^i + P(\hat n)^i{}_j B^j$

~~$= (\delta^i{}_j + \epsilon \hat n^i \hat n_j) B^j$~~

$= (\epsilon \hat n^i \hat n_j + P^i{}_j) B^j$

insert back in F (next page)

$\boxed{11.6}$
3

$P(\hat{n})\mathbf{X} = \mathbf{X}$ for "surface vector"

$$g(\mathbf{X},Y) = g(P(\hat{n})\mathbf{X}, P(\hat{n})Y)$$

$$= \underbrace{g_{ij}\, P^i(\hat{n})_m\, P(\hat{n})^j{}_n}\ \mathbf{X}^m Y^n$$

$^{(2)}g_{mn}$ projected metric gives inner products on tangent plane.

$$\eta(\mathbf{X},Y,Z) = \eta(\underbrace{P(\mathbf{X}), P(Y),}_{} P(Z)) = 0$$

only 2 ind vectors 3-form on 2-dim vector space is zero!

$\eta(\mathbf{X}, \downarrow, \downarrow)$

↑ project on first index

zero if last 2 inputs are surface vectors

$$F = B^k \eta_{k|ij|}\, dx^i \wedge dx^j$$

$$= B^m \underset{\sim}{\delta^k{}_m}\, \eta_{k|ij|}\, dx^i \wedge dx^j = B^m(\epsilon \hat{n}^k \hat{n}_m + P^k{}_{m})\, \eta_{k|ij|}\, \wedge\ dx^i \wedge dx^j$$

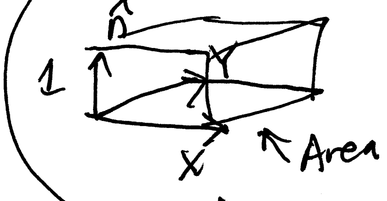
$$\underbrace{\epsilon(B^m \hat{n}_m)\,\hat{n}^k}_{} \qquad \eta_{k|ij|} \wedge dx^i \wedge dx^j$$

$$B^\perp \hat{n}^k$$

$(2)\ \eta_{i|ij|}\, dx^i \wedge dx^j$

unit surface area 2-form

volume of $\hat{n}, X, Y$ $\Rightarrow$ $\eta(\hat{n}, X, Y)$



$V = 1 \cdot$ Area

$\underset{\sim}{\ }$ ↳ normal parts dont contribute

= area of X, Y parallelogram

$$\int_\Sigma F = \int_U \psi^* F\ ~~du\,dv~~{}^2 = \int \psi^*(B^{\perp}\,{}^{(2)}\eta) + 0$$

$\uparrow \psi^*(P\eta) = 0$

$\boxed{11.6}$
4

$$\Psi^* F = \Psi^*(B^{\perp} \, {}^{(2)}\eta_{ij} dx^i \wedge dx^j)$$

$$\Psi^* g = g_{\alpha\beta} du^\alpha \otimes du^\beta$$

pullback = metric on surface

pullback $= B^{\perp} \circ \Psi \, \underbrace{(\Psi^* \, {}^{(2)}\eta_{[ij]}}_{\text{"}dS\text{"}} \longrightarrow$ volume 2-form

"dS" $= \dots = |\det(g_{\alpha\beta})|^{1/2} du^1 \wedge du^2$

$$= B^{\perp} dS$$

$$= \epsilon \vec{B} \cdot \underbrace{\hat{n} \, dS}_{d\vec{S}} = \epsilon \vec{B} \cdot d\vec{S} \quad , \quad d\vec{S}^{\,i} \doteq \hat{n}^i dS$$

$$= \hat{n}^i \, {}^{(2)} \eta$$

$$= \hat{n}^i \hat{n}^k \eta_{k|ij|} dx^i \wedge dx^j$$

<u>Euclidean $\mathbb{R}^3, \epsilon = 1$:</u>

$$F = B^1 dx^2 \wedge dx^3 + B^2 dx^3 \wedge dx^1 + B^3 dx^1 \wedge dx^2$$

$$= \vec{B} \cdot d\vec{S} \longrightarrow d\vec{S} = \langle dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2 \rangle$$

vector valued 2-form whose pullback to a surface yields $\hat{n} \, dS$

<u>interpretation</u>

$$\int_\Sigma F = \int B^{\perp} dS$$

= integral of normal component of vector field $B^\#$ corresponding to 1-form B with respect to differential of surface area

normal component instead of tangential component because of duality operation — switches from tangential directions to normal directions

$\boxed{11.6}$

5

$$\int_{\underset{\uparrow}{\underset{\text{p-surface}}{\Sigma}}} \overset{\text{p-form}}{\overset{\downarrow}{F}} = \int \text{tangential components of } F \text{ wrt } dS_{(p)}$$

$$= \int \text{normal components of } {}^*F \text{ wrt } dS_{(p)}$$

$\frac{n}{2} < p < n$ then $\underbrace{n-p}_{\text{dual has}} < p$

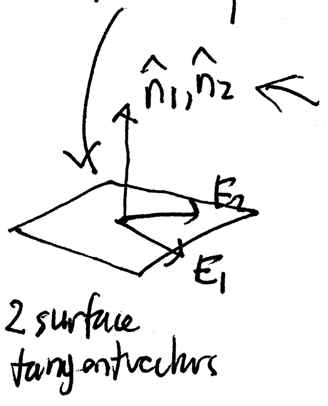
fewer components than T

more economical

if $\frac{n}{2} = p$ (n even) then $n-p=p$ same # indices

$\underline{n=4}$

B	$^*B=F$
$p=0$	$n-p=4 \rightarrow$ integral of 4-form $\rightarrow$ integral dual scalar d^4V
$p=1$	$n-p=3 \rightarrow$ integral of 3-form $\rightarrow$ integral dual vector d^3V (normal component)
$p=2$	$n-p=2 \rightarrow$ integral of 2-form $\rightarrow$ integral of dual 2-form $d2V$ (no advantage)



2 orthogonal unit normals

$$\hat n_1 \wedge \hat n_2 \wedge \hat E_1 \wedge \hat E_2 = \underset{>0}{k}\, \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial_4$$

$$\underbrace{F(\hat n_1, \hat n_2, \ , \ )}_{\text{tangential 2-form}}$$

pullsback to multiple of $du^1 \wedge du^2$
integrate this scalar multiple

$\boxed{11.6}$ what next? 2 days!

6

1) exterior derivative of p-forms (needs no metric) (11.7)

$F \rightarrow dF$ (p+1)-form

↳ can re-express in terms of metric if present (11.8)

$\boxed{\mathbb{R}^3}$ $p=0$: $f \rightarrow \vec{\nabla} f$

$p=1$: $\vec{X} \Leftrightarrow \text{curl}\,\vec{X} = \vec{\nabla}\times\vec{X}$

$p=2$: $*F = \vec{X} \rightarrow$ ~~[illegible] dF~~ $\Leftarrow$ div $\vec{X}$

2) Stoke's Thm on n-dim space (11.9 - 11.20)

$$\int_\Sigma dF = \int_{\partial\Sigma} F$$

without metric

Σ ↓ (p+1)surface

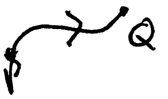
dF ↑ (p+1)-fm

$\partial\Sigma$ ↑ p-surface boundary of Σ

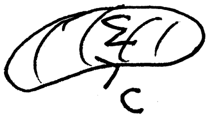
F ↑ p-form

↓ ↓ translate into equivalent metric expressoons on each side

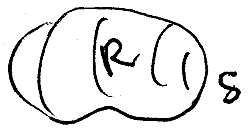
$\boxed{\mathbb{R}^3}$



$$\int_C \vec{\nabla}\phi \cdot d\vec{s} = \phi(Q) - \phi(P)$$



$$\int_\Sigma (\nabla\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{s}$$



$$\int_R \vec{\nabla}\cdot\vec{F}\, dV = \int_S \vec{F}\cdot d\vec{S}$$

main ideas in class — rest read in book if you are still with me

examples (11.11 - 11.12)

$\boxed{11.7-.8}$ **exterior derivative d** (coordinate frame)

1

0-form: $f \rightarrow df = \partial_i f\, dx^i = \boxed{f_{,i}\, dx^i}$ 1-form.

1-form $\sigma = \sigma_i dx^i \rightarrow d\sigma = d\sigma_i \wedge dx^i = \partial_j \sigma_i\, dx^j \wedge dx^i$

$$= \partial_{[j} \sigma_{i]}\, dx^j \wedge dx^i$$

$$= \tfrac{1}{2} \underbrace{(\partial_j \sigma_i - \partial_i \sigma_j)}\, dx^j \wedge dx^i$$

$$[d\sigma]_{ji} = \boxed{2\, \partial_{[j} \sigma_{i]}} = \partial_j \sigma_i - \partial_i \sigma_j$$

no factor

2-form $F = \tfrac{1}{2} F_{ij}\, dx^i \wedge dx^j \rightarrow dF = \tfrac{1}{2} dF_{ij} \wedge dx^i \wedge dx^j$

$$= \tfrac{1}{2} \partial_k F_{ij}\, dx^k \wedge dx^i \wedge dx^j$$

$$= \tfrac{1}{2} \partial_{[k} F_{ij]}\, dx^k \wedge dx^i \wedge dx^j$$

$$= \tfrac{1}{3!} [dF]_{kij}\, dx^k \wedge dx^i \wedge dx^j$$

$$\therefore [dF]_{kij} = \boxed{3\, \partial_{[k} F_{ij]}}$$

$$= 3 \tfrac{1}{3!} \begin{bmatrix} \partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki} \\ - \partial_k F_{ji} - \partial_i F_{kj} - \partial_j F_{ik} \end{bmatrix}$$

easy to remember cyclic sum

$$= \partial_k F_{ij} + \partial_i F_{jk} + \partial_j F_{ki}$$

~~$= \partial_k F_{ij} - \partial_i F_{kj} + \partial_j$~~ $= \partial_k F_{ij} - \partial_i F_{kj} + \partial_j F_{ki}$

no factor this generalizes

Keep order, bring each one successively to first position — alternate sign

p-form $T = \tfrac{1}{p!} T_{i_1 \ldots i_p}\, dx^{i_1} \wedge \ldots \wedge dx^{i_p}$

$$\rightarrow dT = \tfrac{1}{p!} dT_{i_1 \ldots i_p} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_p}$$

$$\boxed{[dT]_{i_1 \ldots i_{p+1}} = (p+1)\, \partial_{[i_1} F_{i_2 \ldots i_{p+1}]}}$$

$$= \partial_{i_1} F_{i_2 i_3 \ldots i_{p+1}} - \partial_{i_2} F_{i_1 i_3 \ldots i_{p+1}} + \partial_{i_3} F_{i_1 i_2 \ldots i_{p+1}} - \ldots$$

no factor

$$[d^2 T]_{i_1 \ldots i_{p+2}} = (p+1)(p+2)?\, \underbrace{\partial_{[i_1} \partial_{i_2}}_{\rightarrow 0} F_{i_3 \ldots i_{p+2}]} = 0$$

$$\boxed{d^2 = 0}$$

compare!

$$\nabla_{i_1} T_{i_2 i_3} = \underbrace{\partial_{i_1} T_{i_2 i_3} - \Gamma^k{}_{i_1 i_2} T_{\overset{①}{k} i_3}} - \Gamma^k{}_{i_1 i_3} T_{i_2 \overset{②}{k}}$$

$$\nabla_{[i_1} T_{i_2 i_3]} = \partial_{[i_1} T_{i_2 i_3]} - \Gamma^k{}_{\underbrace{[i_1 i_2}_{\to 0}} T_{|k| i_3]} - \Gamma^k{}_{\underbrace{[i_1 i_3}_{\to 0}} T_{i_2]k}$$

lower coord comps symmetric.

or $T_{[i_2 i_3 ; i_1]} = T_{[i_2 i_3 , i_1]}$ "comma to semicolon rule"
extra connection terms cancel
für symmetric connection

tensor-valued differential forms

tensor with group of antisymmetric indices + extra indices

$L^i{}_j$: $L^i = L^i{}_j dx^j$ "vector-valued 1-form"

$$\left[L = L^i{}_j \frac{\partial}{\partial x^i} \otimes dx^j = \frac{\partial}{\partial x^i} \otimes L^i{}_j dx^j = \frac{\partial}{\partial x^i} \otimes L^i \right]$$

$$d(L_j dx^j) = 2\nabla_{[k} L_{j]} dx^k \wedge dx^j$$

$$D(L^i{}_j dx^j) = 2\nabla_{[k} T^i{}_{j]} dx^k \wedge dx^j$$

$$= 2\Big(\underbrace{\partial_{[k} T^i{}_{j]}}_{dT^i} + \Gamma^i{}_{[k|m|} T^m{}_{j]} - \overset{0}{\cancel{\Gamma^m{}_{[kj]}}} T^i{}_m\Big) dx^k \wedge dx^j$$

$$+ \underbrace{\Gamma^i{}_{km} dx^k}_{\omega^i{}_m \wedge T^m} \wedge (T^m{}_j dx^j)$$

$$= dT^i + \omega^i{}_m \wedge T^m$$

L_{ij}: $L_i = L_{ij} dx^j$ … $DL_i = dL_i - \omega^m{}_i \wedge L_m$
etc…

$\Omega^m{}_n = \frac{1}{2} R^m{}_{n\check{i}\check{j}} dx^i \wedge dx^j$ $\binom{1}{1}$-tensorvalued 2-form

$$D\Omega^m{}_n = d\Omega^m{}_n + \omega^m{}_p \wedge \Omega^p{}_n - \omega^p{}_{\cancel{m}} \wedge \Omega^m{}_{\cancel{p}}$$

2-form

$$-\Omega^m{}_p \wedge \omega^p{}_n$$

$$\overset{-\ \ +}{dx^k \wedge dx^i \wedge dx^j} = dx^i \wedge dx^j \wedge dx^k$$

$$\boxed{D\underline{\Omega} = d\underline{\Omega} + \underline{\omega} \wedge \underline{\Omega} - \underline{\Omega} \wedge \underline{\omega}}$$

$\boxed{11\text{-}7\text{-}08}$

3

$$R^m{}_{n\underbrace{ij}} = \underbrace{\partial_i \Gamma^m}{}_{jn} - \underbrace{\partial_j \Gamma^m}{}_{in} + \Gamma^m{}_{\underbrace{ip}\Gamma^p{}_j}{}_n - \Gamma^m{}_{\underbrace{jp}\Gamma^p{}_i}{}_n$$

$$\downarrow \qquad = 2\,\partial_{[i}\Gamma^m{}_{j]n} + 2\Gamma^m{}_{[i|p|}\Gamma^p{}_{j]n} \qquad\qquad \Gamma^m{}_{\underbrace{j}n}\,d\underbrace{x^j} = \omega^m{}_n$$

$$\boxed{\Omega^m{}_n = d\omega^n{}_n + \omega^m{}_p \wedge \omega^p{}_n}$$

$$\underline{\Omega} = d\underline{\omega} + \underline{\omega}\wedge\underline{\omega} \qquad \text{(matrix multiplication implied)}$$

previous page

$$D\underline{\Omega} = d\underline{\Omega} + \underline{\omega}\wedge\underline{\Omega} - \underline{\Omega}\wedge\underline{\omega}$$

$$= d(d\underline{\omega} + \underline{\omega}\wedge\underline{\omega}) + \underline{\omega}\wedge\underline{\Omega} - \underline{\Omega}\wedge\underline{\omega}$$

$$= \underbrace{d^2\underline{\omega}}_{=0} + d(\underline{\omega}\wedge\underline{\omega}) + \underline{\omega}\wedge\underline{\Omega} - \underline{\Omega}\wedge\underline{\omega}$$

$$= (d\omega\wedge\omega - \omega\wedge d\omega) + \underline{\omega}\wedge\underline{\Omega} - \underline{\Omega}\wedge\underline{\omega}$$

wedge product rule?

$$= (\underline{\Omega} - \underline{\omega}\wedge\underline{\omega})\wedge\underline{\omega} + \underline{\omega}\wedge\underline{\Omega}$$
$$- \underline{\omega}\wedge(\underline{\Omega} - \underline{\omega}\wedge\underline{\omega}) - \underline{\Omega}\wedge\underline{\omega}$$

$$= \cancelto{0}{\underline{\Omega}\wedge\underline{\omega}} - \cancelto{0}{\underline{\omega}\wedge\underline{\omega}\wedge\underline{\omega}} + \cancelto{0}{\underline{\omega}\wedge\underline{\Omega}}$$
$$- \cancel{\underline{\omega}\wedge\underline{\Omega}} + \cancel{\underline{\omega}\wedge\underline{\omega}\wedge\underline{\omega}} - \cancel{\underline{\Omega}\wedge\underline{\omega}}$$

ex:

$$d(\underset{p=1}{S}\wedge\underset{q=1}{T}) = d(S_i dx^i \wedge T_j dx^j)$$

$$= d(S_i T_j\, dx^i \wedge dx^j)$$

$$= d(S_i T_j)\wedge dx^i \wedge dx^j$$

$$= \partial_k(S_i T_j)\, dx^k \wedge dx^i \wedge dx^j$$

$$= [\underbrace{(\partial_k S_i)}\, T_j + S_i(\partial_k T_j)]\, \underbrace{dx^k \wedge dx^i}\wedge dx^j$$

switch order!

$$= (\partial_k S_i\, dx^k \wedge dx^i)\wedge(T_j dx^j) + (-1)\, S_i dx^i \wedge (\partial_k T_j\, dx^k \wedge dx^j)$$

$$= dS\wedge T - S\wedge dT$$

$$d(\underset{p}{S}\wedge\underset{q}{T}) = dS\wedge T + \underbrace{(-1)^p}\, S\wedge dT$$

\# transpositions to reorder as above

$$0 = D\underline{\Omega} \rightarrow \boxed{\begin{array}{l} R^m{}_{n[ij;k]} = 0 \\ R^m{}_{nij;k} + R^m{}_{njk;i} + R^m{}_{nki;j} = 0 \end{array}}$$

cyclic sum

Bianchi identities of the second kind

$\boxed{11.7\text{-}.8}$

4

$$\delta^i_m\left[R^m{}_{\underbrace{ni}_{1\ 3}j;k} + R^m{}_{njk;i} + R^m{}_{nki;j} = 0\right]$$ contract $\overset{m}{i}$

$$\underbrace{R^i{}_{nij;k}}_{R_{nj;k}} + R^i{}_{njk;i} + R^i{}_{\underbrace{nk}i;j} = 0$$

$$-R^i{}_{nik;j}$$

$$-R_{nk;j}$$

$$g^{nk}\left[R_{nj;k} - R_{nk;j} + R^i{}_{njk;i} = 0\right]$$ contract $n\ k$

$$R^k{}_{j;k} - \underbrace{g^{nk}R_{nk;j}}_{(g^{nk}R_{nk})_{;j}} + \underbrace{g^{nk}R^i{}_{n\underbrace{jk}_{2\ 4};i}} = 0$$

$$R^{ik}{}_{jk} = R^{ki}{}_{kj} = R^i{}_j$$

$$= R^k{}_{j;k} - R_{;j} + \underbrace{R^i{}_{j;i}}_{R^k{}_{j;k}}$$

$$= 2R^k{}_{j;k} - \delta^k_j R_{;k}$$

$$= 2\left[\underbrace{R^k{}_j - \tfrac{1}{2}\delta^k_j R}\right]_{;k} \qquad = 2G^k{}_{j;k} = 0$$

$\equiv G^k{}_j$ Einstein tensor

$$\boxed{G^{jk}{}_{;k} = 0}$$

extends divergence to tensor with extra indices

divergence

$$\frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}$$

$$= \frac{\partial F^1}{\partial x^1} + \frac{\partial F^2}{\partial x^2} + \frac{\partial F^3}{\partial x^3}$$

$$= F^i{}_{,i} \rightarrow F^i{}_{;i}.$$

flat R3 cartesian coords · any coords

$$\boxed{\operatorname{div} F \equiv F^i{}_{;i}}$$ 1 index tensor

↑ Gauss's law in multivariable calc

$\boxed{\text{Zero divergence}}$

↓

key for Einstein equations of general relativity

$\boxed{11.7\text{-}.8}$

5

$\mathbb{R}^3$ in any coordinates metric g_{ij} + d $\longrightarrow$

$p=0$ $\quad df = \partial_i f\, dx^i$

$$(df)^\# = g^{ij}\partial_j f \frac{\partial}{\partial x^i} = \boxed{\vec{\nabla} f = \text{grad}\, f = (df)^\#}$$

$p=1$ $\quad \sigma = \sigma_j\, dx^j$

$$d\sigma = \partial_{[i}\sigma_{j]}\, dx^i \wedge dx^j = \frac{1}{2}\big(\underbrace{2\partial_{[i}\sigma_{j]}}_{(d\sigma)_{ij}}\big)\, dx^i \wedge dx^j$$

$$(*d\sigma)_k = \frac{1}{2}\eta_k{}^{ij}(d\sigma)_{ij} = \eta_k{}^{ij}\partial_{[i}\sigma_{j]} = \eta_k{}^{ij}\partial_i\sigma_j$$

$$(*d\sigma)^\# = \underbrace{\eta^{kij}\partial_i\sigma_j}_{\epsilon^{kij}\partial_i\sigma_j} \frac{\partial}{\partial x^k} \longrightarrow = \eta^{kij}\nabla_i\sigma_j = \vec{\nabla}\times X$$

cartesian coords $\quad \epsilon^{kij}\partial_i\sigma_j$

$(\text{curl}\, \underbrace{\sigma^\#}_{\equiv X})^k$

$$\boxed{\text{curl}\, X = *dX^\flat}$$

$$\overset{|||}{\boxed{\vec{\nabla}\times X}} \leftarrow \eta^i{}_{jk}\nabla^j X^k\partial_i$$

equivalent ib d

$p=2$ $\quad dF_{kij} = 3\partial_{[k}F_{ij]} = 3\nabla_{[k}F_{ij]}$

$$*dF = \frac{1}{3!}\eta^{kij}\, dF_{kij} = \frac{1}{3!}\left(\eta^{kij}\, 3\nabla_{[k}F_{ij]}\right)$$

$$= \frac{1}{2}\eta^{kij}\nabla_k \underbrace{F_{ij}}$$

$$F = *B^\flat \nearrow \eta_{ijm}B^m$$

$$= \frac{1}{2}\underbrace{\eta^{kij}\eta_{ijm}}_{2\delta^k{}_m}\nabla_k B^m$$

$$= \nabla_k B^k = B^k{}_{;k} \equiv \text{div}\, B$$

$$\boxed{*d*B^\flat = \text{div}\, B = \vec{\nabla}\cdot\vec{B}}$$

$\uparrow$

$g_{ij}\nabla^i B^j$

valid for any metric on $\mathbb{R}^3$ — extend flat $\mathbb{R}^3$ geometry to curved 3d spaces.

$$*\underbrace{d\underbrace{*\underbrace{S}_{p}}_{n-p}}_{n-p+1}$$

$n-(n-p+1) = p-1$

d : p-form $\to$ $(p+1)$ form

$*d*$: p-form $\to$ $(p-1)$ form $\sim \delta$

$(d\delta + \delta d)\underset{p\text{-form}}{S} \to$ p-form $\sim \vec{\nabla}\cdot\vec{\nabla}S$ $\boxed{\text{Laplacian}}$

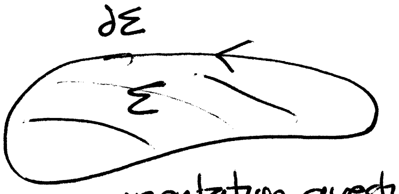
$\boxed{11.7\text{–}.8}$

6

Stokes' Thm:
$$\int_{\underset{p\text{ dim}}{\partial\Sigma}} \underset{\uparrow\ p\text{-form}}{\sigma} = \int_{\underset{p+1\text{ dim}}{\Sigma}} \underset{\uparrow\ p+1\text{ fm}}{d\sigma}$$

$\partial\Sigma$ = boundary Σ

$\boxed{p=1}$
$$\int_{\underset{\text{curve}}{\partial\Sigma}} \sigma = \int_{\underset{\text{2-surface}}{\Sigma}} \underset{\text{2-fm}}{d\sigma}$$



orientation question left to discuss

$$= \int_\Sigma (*d\sigma)^\sharp \cdot d\vec{S}$$

$$= \int \sigma^\sharp \cdot d\vec{s}$$

~~[illegible]~~

$$\int_{\underset{\text{curve}}{\partial\Sigma}} X \cdot d\vec{s} = \int_{\underset{\text{surface}}{\Sigma}} \text{curl}\, X \cdot d\vec{S}$$

$\int_\Sigma F = \int_\Sigma B \cdot d\vec{S}$

$F = *B^\flat$

$B^\flat = *F$

usual $\boxed{\text{Stokes' Thm}}$

$\boxed{p=2}$
$$\int_{\underset{\text{2-surface}}{\partial\Sigma}} \sigma = \int_{\underset{\text{3 region}}{\Sigma}} \underset{\text{3-form}}{d\sigma} \leftarrow \underset{\text{2-fm}}{\sigma} = * \underset{\text{1-fm}}{B^\flat}$$

$= \quad \sigma = *B^\flat \qquad = \quad d*B^\flat$

$$\int_{\partial\Sigma} B \cdot d\vec{S} = \int_\Sigma \underset{3\text{ fm}}{d*B^\flat} = \int_\Sigma (*d*B^\flat)\,\eta = \int_\Sigma \text{div}\, B \; dV$$

S = 3-form

$*S$ = 0-form

$S = (*S)\eta$ (3 fm)

usual $\boxed{\text{Gauss's law}}$

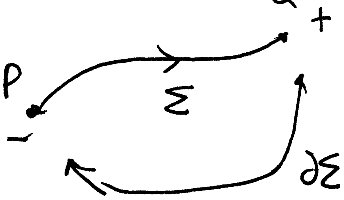
proof generalizes same proof as Green's Thm in plane — just have to adapt notation to make it go through

$\boxed{11.7-18}$

7

p=0 $\quad \int_{\partial\Sigma} \sigma = \int_{\Sigma} d\sigma$

$$\overset{\shortparallel}{\sigma(Q)-\sigma(P)} = \overset{\shortparallel}{\int_{\Sigma} \vec{\nabla}\sigma \cdot d\vec{s}}$$



∂Σ — 2 points oriented by −,+ signs

"induced by orientation of curve"

Σ has orientation
∂Σ has orientation
needed to integrate over on LHS/RHS Stokes' Thm.

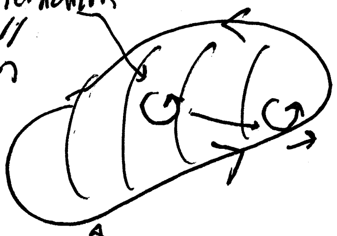
BUT works only if signs correlated:

→ assign orientation to Σ
→ imply "induced orientation" for boundary ∂Σ

---

next dimension:

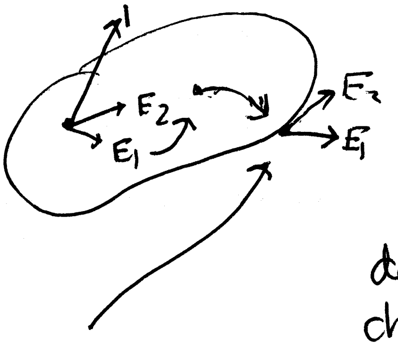
surface Σ orientation
//
circulation sense
↻



bring to boundary ∂Σ gives induced orientation (direction) for curve.

↑ ∂Σ = C

↑
can also be specified by right hand rule from picking a side of tangent plane.



if metric around — side of tangent plane determined by choice of unit normal for Σ

counterclockwise screw sense looking down from outside closed surface

bring to edge so E_1 points off while E_2 is tangent to boundary.

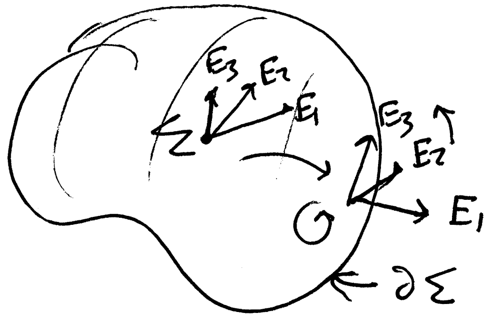
E_2 gives "induced orientation" to ∂Σ

$\boxed{11.7-8}$

8

3-d Σ

region in $\mathbb{R}^3$ with right handed frame



bring to boundary so that E_1 is off $\partial\Sigma$ tan plane by $\underbrace{E_2, E_3}$ span tan plane.

induced orientation

specified either internally (screw sense) or externally (side of tan plane)

give counterclockwise screw sense seen from outside

correlated with outside of region

E_1 points out of region

If metric present → outer normal orients boundary surface

p^{+1}-region Σ → p-dim $\partial\Sigma$

$E_1, \ldots, E_{p+1}$ oriented frame → bring to edge so $E_2 \ldots E_{p+1}$ span p-tanplane but E_1 sticks off $\partial\Sigma$

then $\{E_2, \ldots, E_{p+1}\}$ determine the internal induced orientations

E_1 the external induced orientation

locked together:

$$\text{so } \underset{\underset{\text{external orientation } \to \exists \text{ for } \partial\Sigma}{\uparrow}}{E_1} \wedge (\underbrace{E_2 \wedge \ldots \wedge E_{p+1}}_{\text{internal orientation}}) = \underbrace{E_1 \wedge \ldots E_{p+1}}_{\text{oriented on } \Sigma}$$

$\underbrace{\text{external orientation} \to \exists \text{ for } \partial\Sigma}_{\text{induced orientation on boundary}}$

$\boxed{11,7\text{-}8}$
9

Lorentz case:
must be carefull of <u>signs</u>, timelike / spacelike etc.

just have to look at each particular case
& apply general rules. etc.
in BOOK many worked examples verifying
different p-form cases in different $\mathbb{R}^n$ spaces
with Euclidean or Minkowski metrics

11. end

adjoints

thelastlecture?

scalar product : $\langle \underset{\substack{\uparrow \\ V^* \\ \text{dual space}}}{\sigma}, \underset{\substack{\uparrow \\ V \\ \text{vector space}}}{X} \rangle = \sigma_i X^i$ natural pairing

lineartransformation
$A: V \to V$

$$\langle \sigma, AX \rangle = \sigma_i (A^i{}_j X^j) \\ = (\sigma_i A^i{}_j) X^j$$

$V^* =$ vectorspace

$$\sigma_j \to \sigma_i A^i{}_j = A^i{}_j \sigma_i$$

if components of "vector" on V^*

transposeof A = matrixof lineartrans $A^T : V^* \to V^*$

transpose map

$$\underset{\text{row}}{\sigma^T} \to \sigma^T A = (A^T \sigma)^T$$

$$\sigma \to A^T \underset{\substack{\uparrow \\ \text{column vector}}}{\sigma}$$

real inner product on real V

symmetric bilinear : $\langle Y, X \rangle = g_{ij} Y^i X^j$ $\quad$ ($g_{ij} = g_{ji}$ symmetric)

$$\langle Y, AX \rangle = \underbrace{g_{ij} Y^i A^j{}_k}X^k = Y^i \underset{\substack{\uparrow \\ A_i{}^m g_{mk}}}{A_{ik}} X^k$$

$$= g_{mk} A_i{}^m Y^i X^k$$

$$= \langle A^+ Y, X \rangle$$

"adjoint"

$$(A^+ Y)^m = \overbrace{A_i{}^m Y^i}^{\text{transpose}} \\ = (\underline{g}\underline{A}\underline{g}^{-1})_i{}^m Y^i$$

transpose plusindex shifting

$$\left\{ \begin{aligned} \underline{A}^+ &= (\underline{g}\underline{A}\underline{g}^{-1})^T = (\underline{g}^{-T})^+ \underline{A}^+ \underline{g}^T \\ &= \underline{g}^{-1} \underline{A}^T \underline{g} \end{aligned} \right.$$

no differenceif $\underline{g} = \underline{I}$
Euclidean inner product

complex vectorspace with real self-inner product

$\langle Y, X\rangle = g_{ij}\bar{Y}^i X^j$ ← complex conjugate on one slot:

$$\langle X, X\rangle = \boxed{g_{ij}}\,\bar{X}^i X^j \overset{\text{real}}{=} \overline{\langle X, X\rangle}$$

$$= \overline{g_{ij}\bar{X}^i X^j}$$

$$= \bar{g}_{ij} X^i \bar{X}^j = \bar{g}_{ij}\bar{X}^j X^i$$

$$= \boxed{\bar{g}_{ji}}\,\bar{X}^i X^j$$

(real probabilities in QM)

sesquilinear:

$\overline{\langle Y, X\rangle} = \langle X, Y\rangle$

$\overline{\langle X, X\rangle} = \langle X, X\rangle$ real

$$g_{ij} = \bar{g}_{ji}$$

$$\underline{g} = \underline{\bar{g}}^T \equiv \text{Hermitian conjugate} \equiv \underline{g}^\dagger$$

↑ $\underline{g}$ is called a Hermitian matrix

repeat

$$\langle Y, AX\rangle = g_{ij}\bar{Y}^i A^j{}_k X^k = \cdots$$

$$= \langle A^\dagger_g Y, X\rangle \qquad \underline{A}^\dagger_g = \underline{g}^{-1}\underline{\bar{A}}^T\underline{g}$$

$$= \underline{g}^{-1}\underline{A}^\dagger\underline{g}$$

↑ adjoint:
flip A from X to Y

if $\underline{g} = \underline{I}$ ordinary Hermitian conjugate

complex fields needed in electromagnetics
quantum mechanics

ASIDE

Lie algebras: generate (real or complex)

orthogonal groups: antisymmetric matrices ⟶ symmetries of innerproducts "bilinear"

unitary groups: antiHermitian matrices ⟶ symmetries of complex conjugate innerproducts "sesquilinear

$$O(3, C) \sim O(3,1)$$

complex orthogonal group

real Lorentz group

isomorphic

$\boxed{\text{11.end}}$
3

$$\langle S, T\rangle \equiv \frac{1}{p!} g^{i_1 j_1}\cdots g^{i_p j_p} S_{i_1\cdots i_p} T_{j_1\cdots j_p}$$

← avoid overcounting

$\boxed{\text{p-form inner product}}$

on space of p-forms <u>at each point</u>

$$\Downarrow$$

$$\underbrace{\underset{p}{S}\wedge\underset{n-p}{{}^*T}}_{n\text{-form}} = \underbrace{\langle S, T\rangle}_{\substack{\text{scalar}\\ \| \\ \text{0-form}}}\eta$$

scalar = ${}^*(S\wedge{}^*T)$ up to sign

product rule

$$d(\underbrace{\underset{p-1}{\alpha}\wedge\underbrace{{}^*\underset{p}{\beta}}_{n-p}}_{n-1}) = \underbrace{\underbrace{d\alpha}_{p}\wedge\underbrace{{}^*\beta}_{n-p}} + (-1)^{p-1}\underbrace{\alpha}_{p-1}\wedge d{}^*\beta$$

n-form

$$= \underbrace{\langle d\alpha, \beta\rangle\eta}_{\text{solve for this}} + \langle\alpha, \underbrace{(-1)^{p-1}d{}^*\beta}_{\equiv -\delta\beta}\rangle$$

$$\langle d\alpha, \beta\rangle\eta = d(\alpha\wedge{}^*\beta) + \langle\alpha, \delta\beta\rangle\eta$$

$$\underbrace{\int_\Sigma \underbrace{\langle d\alpha, \beta\rangle}_{\text{pointwise inner product}}\eta} = \int_{\partial\Sigma} d(\underbrace{\alpha\wedge{}^*\beta}) + \int_\Sigma \langle\alpha, \delta\beta\rangle\eta \leftarrow \boxed{\text{in components "integration by parts"}}$$

global inner product on space of p-form fields:
produces real #
by integrating over whole space

if no boundary (sphere!)
if $\alpha\wedge{}^*\beta$ fixed on boundary (Lagrangian calculation)
or if fields drop off "at ∞"
so that integrals converge ←
THIS TERM GOES TO ZERO

$\boxed{\text{d \& δ are adjoints with respect to this functional inner product.}}$

complex case
"Hilbert space"
for Q.M.

in

$\boxed{11.\text{end}}$
4

electromagnetism : d & δ

Maxwell's eqns $F^{ij}{}_{;j} = 4\pi J^i \leftarrow J_i dx^i = \rho\, dt + \underline{J_\alpha dx^\alpha}$

$^*F^{ij}{}_{;j} = 0$

charge density

$\alpha = 1,2,3$

current density

4-current density

$\downarrow$

translate to d, δ (F_{ij} 2-form)

$-\delta F = 4\pi J^\flat$

$(A = -\phi\, dt + A_\alpha dx^\alpha)$

$dF = 0 \longrightarrow$ if set $F = dA$ (4-vector potential)

then $dF = d^2A = 0$

solve half Maxwell

$-\delta dA = 4\pi J^\flat$

$\underline{-d\delta A + d\delta A = 0}$

note $A \to A + d\Lambda$

$F = dA \to dA + d^2\Lambda = dA = F$

"gauge transformation"

electromagnetic field unchanged.

$$-[\underbrace{\delta d + d\delta}_{\Delta_{deR}}]A \underset{+d\delta A}{\wedge}= 4\pi J^\flat$$

Δ_{deR} deRham Laplacian

$= \nabla^2 A$ Laplacian

$\uparrow$ flatspacetime

$$\boxed{\left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) A^i = ☺}$$

wave equation

$g^{ij} \nabla_i \nabla_j A^k$

$= A^k{}_{;i}{}^{;i}$

$-\Delta_{deR} A + d(\underbrace{\delta A}) = 4\pi J^\flat$ if $= 0 \to$ sourcefree region

$-A^i{}_{;i} \overset{?}{=} 0$ Lorentz gauge condition

if not zero then

$\delta(A + d\Lambda) = 0$

$\downarrow\ \underline{\delta d}\Lambda + \delta A = 0$

$\Delta_{deR}\Lambda = -\delta A$ solve to find new A which satisfies condition

$\downarrow$

$\Delta_{deR} A = 0$

$\nabla^2 A = 0$

wave equation for potential

---

or $d[-\delta F = 4\pi J^\flat]$

$-(d\delta + \delta d) F = 4\pi J^\flat$

$-\Delta_{deR} F = 4\pi J^\flat$ vacuum $= 0 \longrightarrow$ wave equation for field