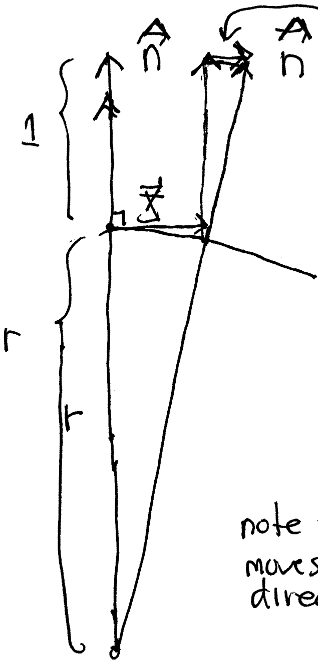
10.1 extrinsic curvature

1

toy example

circle: outward unit normal: $\hat{n}$

surface vector: $\hat{n}\cdot\vec{X}=0$



$\nabla_X \hat{n}$ (change in $\vec{n}$ relative to parallel transported $\vec{n}$)

similar triangles

$$\frac{\vec{X}}{r} = \frac{\nabla_X \hat{n}}{1} \longrightarrow \nabla_X \hat{n} = \frac{1}{r}\vec{X}$$

(lower △) (upper △)

linear transf. of tangential vectors into tangential vectors

note tip of $\vec{n}$ moves in same direction as $\vec{X}$

$$S(\vec{X}) = \frac{1}{r}\vec{X}$$

$$S^i{}_j X^j = \frac{1}{r} X^i = \frac{1}{r}\delta^i_j X^j$$

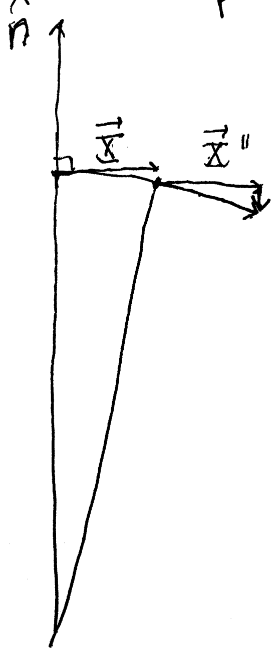
shape tensor

$$S^i{}_j \equiv -K^i{}_j$$

"shape tensor" "extrinsic curvature"

diff geom G.R. jargon

repeat for surface vector $\vec{X}\cdot\hat{n}=0$



$$\frac{|\vec{X}|}{r} = \frac{|\nabla_X \vec{X}|}{|X|}$$

$$|\nabla_X \vec{X}| = \frac{|X|^2}{r}$$

$$\nabla_X \vec{X} = -\frac{\vec{X}\cdot\vec{X}}{r}\hat{n} \quad \text{inward}$$

$$\vec{X}\cdot S(\vec{X}) = \frac{1}{r}\vec{X}\cdot\vec{X}$$

$$X_i S^i{}_j X^j = S_{ij} X^i X^j$$

$$\equiv S^b(X,X)$$

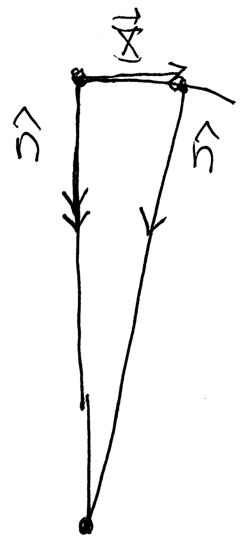
$$= -S^b(X,X)\,\hat{n}$$

symmetric $\begin{cases}(0,2) \text{ tensor } S_{ij} \\ (1,1) \text{ tensor } S^i{}_j\end{cases}$ for surface vectors

describes bending of surface in two ways:

- bending of normal to stay orthogonal
- bending of surface vectors to stay tangent

$\hat{n} \to -\hat{n}$



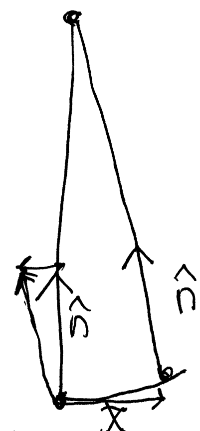
$\nabla_X \hat{n} = S(\vec{X})$

$\downarrow \qquad \downarrow$

$-\hat{n} \quad -S(X)$

overall sign of S not well defined
depends on choice of 2 unit normals

is like



normal tip moves oppositely to X direction

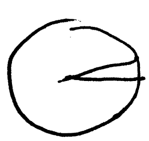
for a given choice it tells you which side of tangent line curve is curving

osculating circle center on same side when
$S(X) = -\frac{1}{r} X$

on opposite side when
$S(X) = \frac{1}{r} X$
(previous page)

Remark: All curves are intrinsically flat!

$g = ds \otimes ds$ in arclength parametrization



$s = r d\theta \to r^2 d\theta^2 = ds^2$

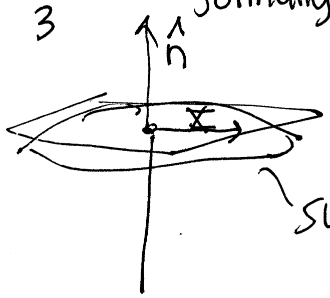
our notion of curvature from the circle is BENDING in larger space = extrinsic curvature

circle = prototype

Intrinsic curvature only starts in 2-dimensions —
sphere = prototype for understanding intrinsic surface curvature

extrinsic curvature on sphere
Comes from circular cross-sections — 1d notion
combine together to quantify 2d bending

formally:



$$\nabla_X (\hat{n}\cdot\hat{n} = 1)$$

$$(\nabla_X \hat{n})\cdot\hat{n} + \hat{n}\cdot(\nabla_X \hat{n}) = 0$$

$$2\hat{n}\cdot\underbrace{(\nabla_X \hat{n})}_{\perp \hat{n}} = 0$$

$\nabla_X \hat{n} = S(X)$ — linear transformation of surface vectors into surface vectors

↑ linear in X

$$\hat{n}^i{}_{;j} X^j = S^i{}_j X^j$$

another surface vector Y ⟶ $Y\cdot\nabla_X\hat{n} = S(X)\cdot Y = Y_i S^i{}_j X^j = S_{ij} Y^i X^j$
$= S^\flat(Y,X)$

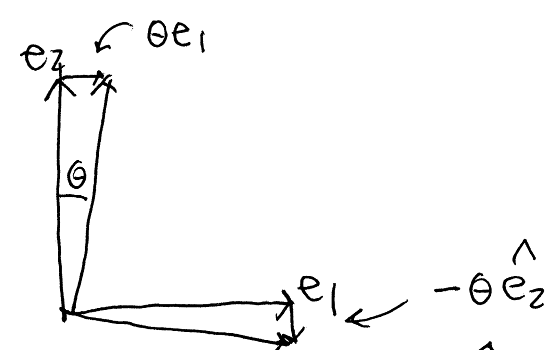
$$\nabla_X (Y\cdot\hat{n} = 0)$$

$$(\nabla_X Y)\cdot\hat{n} + \underbrace{Y\cdot\nabla_X\hat{n}}_{S^\flat(Y,X)} = 0$$

$$S^\flat(Y,X) = -\hat{n}\cdot(\nabla_X Y)$$

derivative of tangential is minus normal

derivative of normal is tangential



symmetric $\binom{0}{2}$ tensor

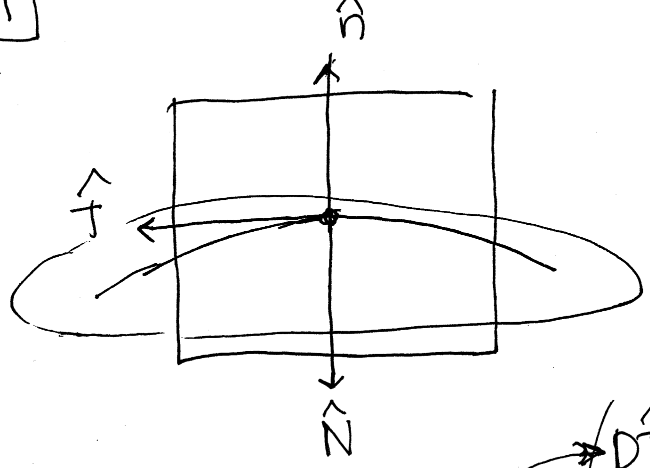
$S^\flat(Y,X) - S^\flat(X,Y) = S_{ij}Y^iX^j - S_{ij}X^iY^j = (S_{ij} - S_{ji})Y^iX^j$

$= -\hat{n}\cdot\nabla_X Y + \hat{n}\,\nabla_Y X$ $\quad\therefore\ \boxed{S_{ij} = S_{ji}}$

$= \hat{n}\cdot(\nabla_Y X - \nabla_X Y)$ ↙ symmetric connection

$= \hat{n}\cdot \underbrace{[X,Y]}_{\text{also surface vector}} = 0$

also surface vector
(express in surface coords — Lie bracket also surface vector)



surface curve with $\hat{T}, \hat{N}, \hat{B}$
obtained by cross-section through surface normal
so curve normal: $\hat{N} = \pm \hat{n}$

flat space
$d = D$

$$\left(\frac{D\hat{T}}{ds} = \nabla_{\hat{T}} \hat{T} \equiv \kappa \hat{N} \right)$$ definition of curve curvature.

$\hat{n} \cdot$

$$\downarrow$$

$$\hat{n} \cdot \nabla_{\hat{T}} \hat{T} = \kappa\, \hat{n} \cdot \hat{N} = \pm \kappa$$

$$\parallel$$

$$-S(\hat{T}, \hat{T}) = -S_{\hat{T}\hat{T}}$$

[sign tells if $\hat{N}, \hat{n}$ parallel (−) or antiparallel (+): $S_{\hat{T}\hat{T}} = -(\hat{n} \cdot \hat{N})\kappa$]

up to sign component of shape tensor along $\hat{T}$ = curve curvature

Shape tensor = machine to produce curvature of curves in surface with direction $\hat{T}$

(S_{ij}) = symmetric matrix − has orthonormal basis of eigenvectors unique if eigenvalues distinct (modulo signs)

$$Y_i [S^i{}_j X^j = \lambda_X X^i] \rightarrow S_{ij} Y^i X^j = \lambda_X X^i Y_i$$

$$X_i [S^i{}_j Y^j = \lambda_Y Y^i] \rightarrow S_{ij} X^i Y^j = \lambda_Y X_i Y^i$$

$$0 = \underbrace{(\lambda_X - \lambda_Y)}_{\neq 0} X_i Y^i \quad \Rightarrow X_i Y^i = 0$$

distinct eigenvalues have orthogonal eigenspaces
if all distinct − 1d eigenspaces → orthogonal → orthonormal frame

(true for hypersurface in $\mathbb{R}^n$)

10.1

5

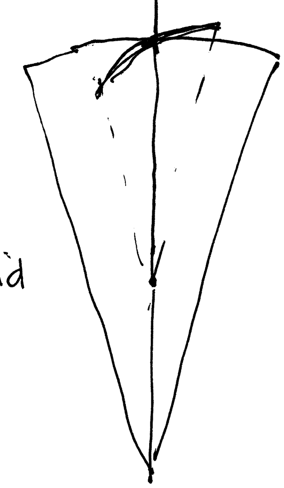
surface in $\mathbb{R}^3$ 2x2 matrix

$$\underline{B}^{-1}\underline{S}\,\underline{B} = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

2 eigenvalues "principal curvatures"

eigenvectors e_1, e_2 principal directions of curvature in 2 orthogonal directions

$\pm\hat{n}$



like ellipsoid

k_1, k_2 same sign

both curve on same side of tangent plane (no intersection with tangent plane) → > 0 positive curvature

$\det \underline{S} = \det(\underline{B}^{-1}\underline{S}\,\underline{B}) = k_1 k_2$

= Gaussian curvature K? $= \det(K^i{}_j)$

like hyperbolic paraboloid.

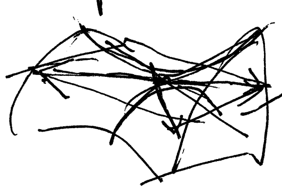
k_1, k_2 opposite signs

on opposite sides of tangent plane

negative curvature < 0

saddle

surface intersects tangent plane in "~~hyperbola~~" "X"



eigenvector, eigenvalue:

$$\nabla_{e_i}\hat{n} = S(e_i) = k_i e_i$$

for each i: ↳ as move along e_i (unit tangent to curve) $\hat{n}$ rotates in plane of e_i & $\hat{n}$

surface normal aligned with curve normal along "lines of curvature"

$k_1 = 0,\ k_2 \neq 0$

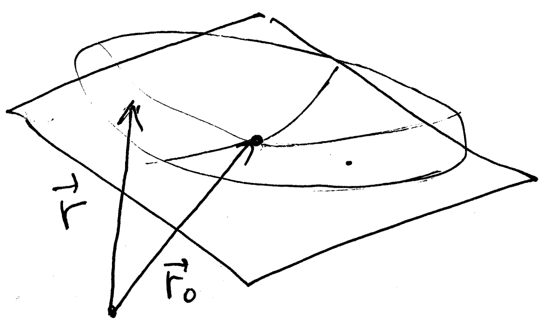
flat in 1 direction, curved in ⊥ direction

$\vec{r} = \vec{r}(u^1, u^2)$ **parametrized surface**

$$\left\{ \begin{array}{l} \vec{r}_1 = \dfrac{\partial \vec{r}}{\partial u^1}(u^1,u^2) \\ \vec{r}_2 = \dfrac{\partial \vec{r}}{\partial u^2}(u^1,u^2) \end{array} \right\}$$

tangents to coord lines in surface – span tangent plane.

$$\left\{ \begin{array}{l} \vec{r}_{11} = \dfrac{\partial^2 r}{\partial u^{1\,2}} \\ \vec{r}_{12} = \vec{r}_{21} = \dfrac{\partial^2 r}{\partial u^1 \partial u^2} \\ \vec{r}_{22} = \dfrac{\partial^2 r}{\partial u^{2\,2}} \end{array} \right.$$



$\vec{r}(u^1_0, u^2_0) \equiv \vec{r}_0$

$u^i - u^i_0 \equiv \Delta u^i$

2nd degree Taylor approximation

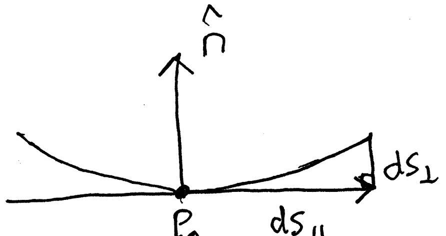
$$\vec{r} - \vec{r}_0 = (\vec{r}_1)_0 \Delta u^1 + (\vec{r}_2)_0 \Delta u^2 \quad \text{\} linear approx}$$

$$+ \frac{1}{2}\Big[+(\vec{r}_{11})_0 (\Delta u^1)^2 + 2(\vec{r}_{12})_0 \Delta u^1 \Delta u^2 + (\vec{r}_{22})_0 (\Delta u^2)^2 \Big] \quad \text{\} quadratic terms}$$

+ higher order terms.

$\Delta u^i \to du^i$

$$ds_\perp = \hat{n} \cdot (\vec{r} - \vec{r}_0) = \frac{1}{2} \underbrace{(\vec{r}_{ab} \cdot \hat{n})}_{S_{ab}} du^a du^b$$



S_{ab} shape tensor in surface coords gives quadratic approximation to surface away from tangent plane

$$ds_\parallel^2 = (\vec{r}_1 du^1 + r_2 du^2) \cdot (\vec{r}_1 du^1 + \vec{r}_2 du^2)$$

$$= g_{ab}\, du^a du^b \qquad a,b = 1,2$$

intrinsic metric

"first fundamental form"
$ds^2 = E du^2 + 2F du dv + G dv^2$

"second fundamental form"
$K du^2 + 2L du dv + M dv^2$

10.4 Geodesic normal coordinates (extrinsic curvature of family of surfaces)

1

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$ spherical coords ($\mathbb{R}^3$) spheres

$$= -dt^2 + t^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$ 3-d cosmology

$$= -d\tau^2 + \tau^2(d\chi^2 + \sinh^2\chi\, d\phi^2)$$ 2-sheet

$$= d\ell^2 + \ell^2(-d\chi^2 + \cosh^2\chi\, d\phi^2)$$ 1-sheet

(2-sheet and 1-sheet: pseudospheres (M^4))

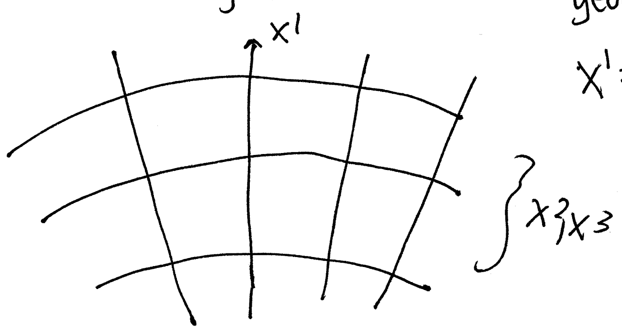
$$= \epsilon (dx^1)^2 + g_{ab}\, dx^a dx^b$$ $a, b = 2,3\ (4)$ — spacetime in 4-d or $\mathbb{R}^4$

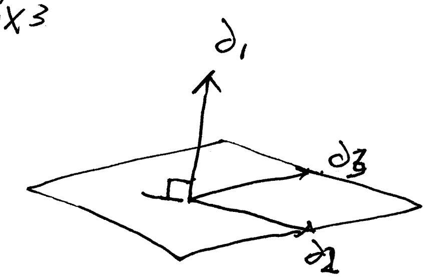
$\epsilon (dx^1)^2$: arclength parametrized geodesics

$g_{ab}\, dx^a dx^b$: orthogonal surfaces

"geodesically parallel"

$x^1 = x^1_0$ surfaces are equidistant (just Δx^1)





orthogonal decomposition of tangent space

$3 = 1 + 2$

conditions

$g_{11} = \epsilon$

$g_{1a} = 0$

g_{ab} whatever (in general) → diagonal above examples

this splitting of tangent space results in splitting of all tensors according to segregating indices as 1 or 2,3

examine extrinsic curvature of each such surface in coord system

(1-parameter family of surfaces)

10.4

2

coord frame:

$e_1 = \partial_1 = \hat{n}$ (unit normal to surfaces $x^1 = x^1_0$)

$e_a = \partial_a$

$\hat{n}\cdot\hat{n} = \epsilon$

$\epsilon\,\hat{n}\cdot\hat{n} = 1$

metric & inverse metric matrices:

$$\underset{\sim}{g} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & g_{11} & g_{12} \\ 0 & g_{21} & g_{22} \end{bmatrix} \qquad \underset{\sim}{g}^{-1} = \begin{bmatrix} \epsilon & 0 & 0 \\ 0 & g^{11} & g^{12} \\ 0 & g^{21} & g^{22} \end{bmatrix}$$

2x2 block ⟷ inverses ⟷ 2x2 block

$g^{11} = \epsilon$

$g^{1a} = 0$

$g^{ab} g_{bc} = \delta^a{}_c$

shape tensor

$\nabla_X \hat{n} = S(X)$ ⟶ $\hat{n}\cdot\nabla_X Y = -S(Y,X)$

$\nabla_{e_a}\hat{n} = S^b{}_a\, e_b$ $\qquad$ $\hat{n}\cdot\nabla_{e_a} e_b = -S_{ab}$

↓

$\nabla_{e_a} e_1 = \Gamma^b{}_{a1}\, e_b$ $\quad\therefore\ S^b{}_a = \Gamma^b{}_{a1} = \Gamma^b{}_{1a}$

$\hat{n}\cdot\left(\nabla_{e_a} e_b = -\epsilon S_{ab}\hat{n} + \Gamma^c{}_{ab}\, e_c\right)$

$= \Gamma^1{}_{ab}\, e_1 + \Gamma^c{}_{ab}\, e_c$

$\epsilon S_{ab} = -\Gamma^1{}_{ab}$

connection components split

$\Gamma_{\gamma\alpha\beta} = \frac{1}{2}(g_{\gamma\alpha,\beta} - g_{\alpha\beta,\gamma} + g_{\beta\gamma,\alpha})$ $\qquad$ $\alpha,\beta,\gamma = 1,2,3$ but

cannot have more than 1 index equal to 1 (",1") $\qquad$ $g_{11} = \epsilon$, $g_{1a} = 0$

same for: ← $\qquad$ $g_{11,1} = 0$, $g_{11,a} = 0$ $\qquad$ $g_{1a,1} = 0$, $g_{1a,b} = 0$

$\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}(g_{\delta\alpha,\beta} - g_{\alpha\beta,\delta} + g_{\beta\delta,\alpha})$

so $\Gamma^1{}_{ab} = -\frac{1}{2} g^{11} g_{ab,1} = -\frac{1}{2}\epsilon g_{ab,1} = -\epsilon S_{ab}$ ⟶ $\boxed{S_{ab} = +\frac{1}{2} g_{ab,1}}$

$\Gamma^c{}_{1b} = \Gamma^c{}_{b1} = \boxed{\frac{1}{2} g^{cd} g_{bd,1} = S^c{}_b}$

$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}(g_{da,b} - g_{ab,d} + g_{bd,a})$ ← connection on surface

spherical coords

$$(g_{ab}) = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad (S_{ab}) = \left(\tfrac{1}{2} g_{ab,1}\right) = r \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

$$= \frac{1}{r} r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} = \frac{1}{r} (g_{ab})$$

$$S^a{}_b = \frac{1}{r} \delta^a{}_b$$

3-cosmology

$$(g_{ab}) = t^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \longrightarrow S_{ab} = \tfrac{1}{2} g_{ab,t} \rightarrow S^a{}_b = \frac{1}{t} \delta^a{}_b$$

↑ time derivative of surface metric

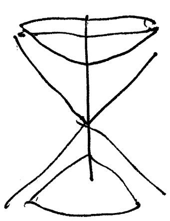
$t = 0$
infinite extrinsic curvature.

metric ~ position in space of dynamical variables	extrinsic curvature shape tensor like velocity

— "dynamics of GR"

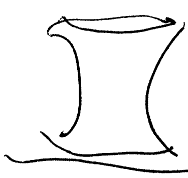
here
"expansion of space"
radius of spheres increasing in time.

1 sheet pseudosphere $\epsilon = -1$,



$$S_{ab} = \frac{1}{\tau} g_{ab} \rightarrow S^a{}_b = \frac{1}{\tau} \delta^a{}_b$$

2 sheet pseudosphere $\epsilon = 1$ $\qquad S^a{}_b = \frac{1}{\ell} \delta^a{}_b$



$$g_{ab} = |X'|^2 g^{(1)}_{ab} \longrightarrow \boxed{S^a{}_b = \frac{1}{|X'|} \delta^a{}_b} \quad \text{in all cases}$$

↑ scales like square of radius/ pseudoradius

↑ unit sphere-pseudosphere

10.4
4

split curvature

$$R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha{}_{\delta\beta} - \partial_\delta \Gamma^\alpha{}_{\gamma\beta} + \Gamma^\alpha{}_{\gamma\epsilon}\Gamma^\epsilon{}_{\delta\beta} - \Gamma^\alpha{}_{\delta\epsilon}\Gamma^\epsilon{}_{\gamma\beta}$$

split by how many indices are 1:

— none:

$$R^a{}_{bcd} = \partial_c \Gamma^a{}_{db} - \partial_d \Gamma^a{}_{cb} + \underbrace{\Gamma^a{}_{c\epsilon}\Gamma^\epsilon{}_{db}}_{} - \underbrace{\Gamma^a{}_{d\epsilon}\Gamma^\epsilon{}_{cb}}_{}$$

$$= \begin{cases} \Gamma^a{}_{ce}\Gamma^e{}_{db} \\ +\Gamma^a{}_{c1}\Gamma^1{}_{db} \end{cases} \qquad = \begin{cases} \Gamma^a{}_{de}\Gamma^e{}_{cb} \\ +\Gamma^a{}_{d1}\Gamma^1{}_{cb} \end{cases}$$

$$= {}^{(2)}R^a{}_{bcd} + \Gamma^a{}_{c1}\Gamma^1{}_{db} - \Gamma^a{}_{d1}\Gamma^1{}_{cb}$$

$$K\,K - K\,K$$

· you finish as exercise

— one:

$$R^1{}_{bcd} = \partial_c \Gamma^1{}_{db} - \partial_d \Gamma^1{}_{cb} + \Gamma^1{}_{c\epsilon}\Gamma^\epsilon{}_{db} - \Gamma^1{}_{d\epsilon}\Gamma^\epsilon{}_{cb}$$

$$\downarrow \qquad \downarrow \qquad\qquad \epsilon \to e \qquad\qquad \epsilon \to e$$

$$\partial K - \partial K \qquad K\Gamma \qquad K\Gamma$$

← at most 1 index 1 on Γ!

↪ $\nabla K - \nabla K$

you finish as exercise

1 in each pair (order doesn't matter — sign only)

— two:

$$R^1{}_{a1b} = \cdots K_{,1} + KK$$

Einstein tensor

$$G^\alpha{}_\beta = R^\alpha{}_\beta - \tfrac{1}{2} R\,\delta^\alpha{}_\beta \quad \leftarrow \quad R^\alpha{}_\beta = R^{\gamma\alpha}{}_{\gamma\beta}, \quad R = R^\alpha{}_\alpha$$

↳ split $G^1{}_b = \ldots$, $G^1{}_1 = \ldots$ } important for understanding G.R.

after splitting first:

$$R^1{}_1 = R^{\gamma 1}{}_{\gamma 1} = R^{c1}{}_{c1}$$

$$R^1{}_b = R^{\gamma 1}{}_{\gamma b} = R^{c1}{}_{cb}$$

$$R^a{}_b = R^{\gamma a}{}_{\gamma b} = R^{1a}{}_{1b} + \underbrace{R^{ca}{}_{cb}}_{{}^{(2)}R^{ca}{}_{cb}}$$

$$R = R^\alpha{}_\alpha = R^1{}_1 + R^a{}_a$$

10.4
5

total curvature constrains intrinsic/extrinsic curvature

result: $R^a{}_{bcd} = {}^{(2)}R^a{}_{bcd} - \epsilon(K^a{}_c K_{bd} - K^a{}_d K_{bc})$ $b,c=1,2$

$R^a{}_{bcd}$: surface components only of space Riemann

${}^{(2)}R^a{}_{bcd}$: total surface Riemann

$\frac{1}{2}{}^{(2)}R^a{}_{bcd}\,dx^c \wedge dx^d - \epsilon\,(K^a{}_c dx^c) \wedge (K_{bd} dx^d)$ (lowered)

$\left(\frac{1}{2}R^a{}_{bcd}\,dx^c \wedge dx^d\right) = {}^{(2)}\Omega - \epsilon\, K \wedge K^\flat$

2x2 matrices

$[K(X)]^a = K^a{}_b X^b = K^a{}_b dx^b(X)$

$\binom{1}{1}$-tensor = linear transformation tensor naturally viewed as vector-valued differential form

↓ flat space (spacetime)

$0 = {}^{(2)}R^a{}_{bcd} - \epsilon(K^a{}_c K_{bd} - K^a{}_d K_{bc})$

${}^{(2)}R^a{}_{bcd} = \epsilon(K^a{}_c K_{bd} - K^a{}_d K_{bc})$

$= \epsilon(S^a{}_c S_{bd} - S^a{}_d S_{bc})$ shape tensor

raise index: $R^{ab}{}_{cd} = \epsilon\, 2 S^a{}_{[c} S^b{}_{d]}$

$a,b=2,3$: only 1 independent component

${}^{(2)}R^{23}{}_{23} = \epsilon(K^2{}_2 K^3{}_3 - K^2{}_3 K^3{}_2)$

$= \epsilon \begin{vmatrix} K^2{}_2 & K^2{}_3 \\ K^3{}_2 & K^3{}_3 \end{vmatrix} = \epsilon \det \underline{K} = \epsilon \det \underline{S} = \epsilon \det(B^{-1} S_d B)$

$\det(K^a{}_b)$ invariant under basis change → $\det S_d = \det\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} = k_1 k_2$

${}^{(2)}R^{23}{}_{23} = \epsilon\, k_1 k_2$

$k_1 k_2$: Gaussian curvature = product principal curvatures

10.4

6

$ds^2 = \epsilon (dx^1)^2 + g_{ab}\, dx^a dx^b \leftrightarrow$ flat space(-time)

$^{(2)}R^{ab}{}_{cd} = \epsilon 2 K^a{}_{[c} K^b{}_{d]} = \frac{\epsilon}{|x^1|^2} 2 \delta^a{}_{[c} \delta^b{}_{d]} = \frac{\epsilon}{|x^1|^2} \delta^{ab}{}_{cd}$

$K^a{}_b = \frac{1}{|x^1|}$

$\frac{\epsilon}{|x^1|^2}$ → 1-ind component = Gaussian curvature up to sign

constant on $x^1 = x^1_0$ surfaces

$\delta^{ab}{}_{cd}$ → generalized Kronecker delta (quadruple scalar product tensor)

has all required symmetries.

"constant curvature" spheres / pseudospheres.

⟸

scalar curvature $^{(2)}R^{ab}{}_{ab} = {}^{(2)}R$ twice Gaussian curvature

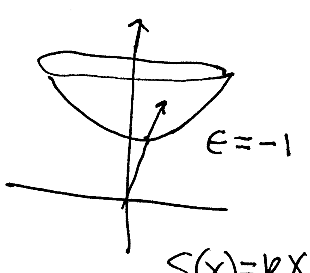
$= \frac{2\epsilon}{|x^1|^2}$

$\epsilon = 1: \ {}^{(2)}R > 0$

$\epsilon = -1: \ {}^{(2)}R < 0$

← intrinsic curvature changes sign with ϵ.

along curvature directions M^3_ϵ

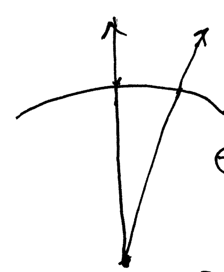


$\epsilon = -1$

$S(x) = kx$, > 0

negative intrinsic curvature

$\mathbb{R}^3$

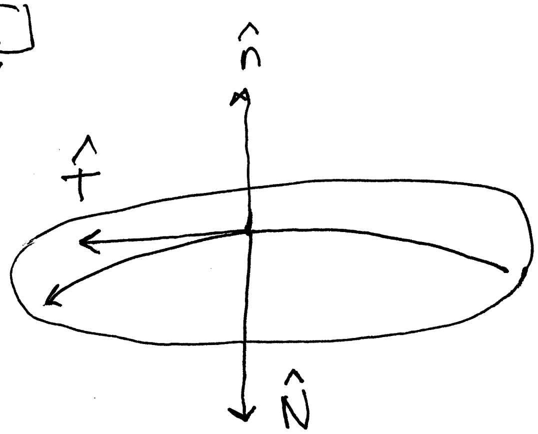


$S(x) = kx$, > 0

$\epsilon = 1$

positive intrinsic curvature

10.1
7



$\frac{D\hat{T}}{ds} = \nabla_{\hat{T}}\hat{T} = \kappa \hat{N}$ surface geo, $\kappa > 0$

$\hat{n} \cdot \nabla_{\hat{T}}\hat{T} = \kappa\, \hat{n}\cdot\hat{N}$

$\parallel$

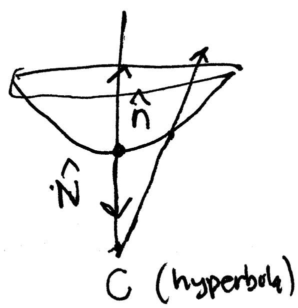
$-S_{\hat{n}}(\hat{T},\hat{T})$

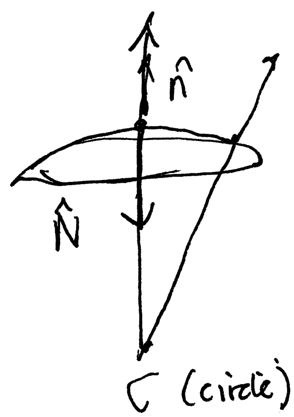
$S_{\hat{n}}(\hat{T},\hat{T}) = -\kappa\, \hat{n}\cdot\hat{N}$

$S_{\hat{n}}(\hat{T},\hat{T}) > 0$

$\hat{n}\cdot\hat{N} < 0$

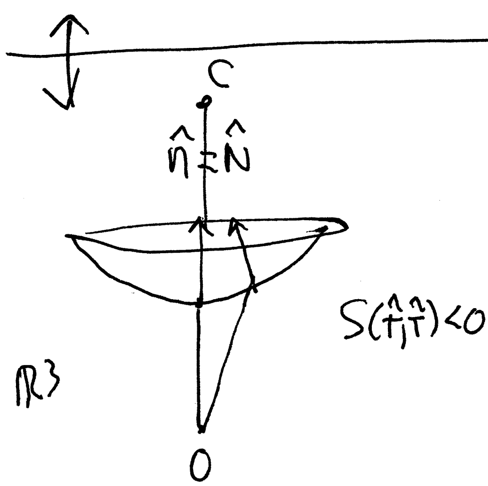
opposite sides: $\hat{n} = -\hat{N}$





M^3 $\epsilon = -1$

$\mathbb{R}^3$ $\epsilon = 1$



$S(\hat{T},\hat{T}) < 0$

$\mathbb{R}^3$

orthogonality leads to opposite behavior of normals