$\boxed{A.3\text{-}4}$

1

$$e_{i'} = e_j B^j{}_{i} \qquad G_{i'j'} = B^m{}_{i'} G_{mn} B^n{}_{j'}$$

$e_{i'}$: ON e_j: ON wrt $\underline{G}$

generalized orthogonal matrix wrt $\underline{G} = \text{diag}(\underbrace{-1,\dots,-1}_{N},\underbrace{1,\dots,1}_{P})$

columns of $\underline{B}$ are new basis

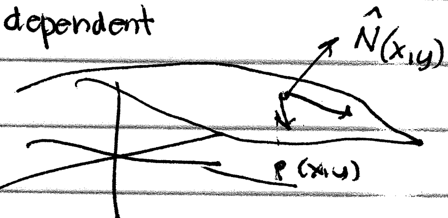
$\underline{B} = \langle \underline{e}_{1'} | \dots | \underline{e}_{n'} \rangle$

$\{e_j\}$ = standard basis of $\mathbb{R}^n$ with this inner product $\underline{G}$.

$\{e_{i'}\}$ = new ON basis which depends on a parameter or on position

$\mathbb{R}^3$ examples

- TNB frame along curve in $\mathbb{R}^3$, functions of parameter t
- $e_{i'}(t)$ = axes fixed in rotating rigid body
  (time dependent)
- $z = f(x,y) \rightarrow$



  $e_{i'} = e_{i'}(x,y)$ on surface.
- $e_{i'} = e_{i'}(x,y,z)$.

matrix $\underline{B}$ = function of variables

how to describe its derivative & interpret meaning?

$\underline{e}' = \underline{e}\,\underline{B} \qquad \underline{e}'\underline{B}^{-1} = \underline{e}$ active transformation of basis

$$d\underline{e}' = \underline{e}\,\underbrace{d\underline{B}}_{\substack{\text{change}\\ \text{relative to}\\ \text{fixed } \underline{e}}} = \underline{e}\,\underline{B}\,\underline{B}^{-1}d\underline{B} = \underline{e}'\underbrace{(\underline{B}^{-1}d\underline{B})}_{\substack{\text{change}\\ \text{relative to}\\ \text{current value of } \underline{e}'}}$$

ON condition : $\underline{G} = \underline{B}^T \underline{G}\,\underline{B}$

$$0 = d\underline{B}^T \underbrace{\underline{G}}_{\underline{B}^{T-1}\underline{B}^T}\underline{B} + \underline{B}^T\underbrace{\underline{G}}_{\underline{B}\,\underline{B}^{-1}}d\underline{B}$$

$$= \underbrace{d\underline{B}^T(\underline{B}^{-1})^T\underbrace{(\underline{B}^T\underline{G}\,\underline{B})}_{\underline{G}=\underline{G}^T}}_{(\underline{G}\,\underline{B}^{-1}d\underline{B})^T} + \underbrace{(\underline{B}^T\underline{G}\,\underline{B})}_{\underline{G}}\underline{B}^{-1}d\underline{B}$$

$$(\underline{G}\,\underline{B}^{-1}d\underline{B})^T + \underline{G}\,\underline{B}^{-1}d\underline{B}$$

$$(\underline{B}^{-1}d\underline{B})_{ji} + (\underline{B}^{-1}d\underline{B})_{ij} = 0$$

dim $n=1$:
$B^{-1}dB$
$= d\ln B$
logarithmic derivative

$\underline{B}^{-1}d\underline{B}$ is antisymmetric when index lowered on left ∴ $\boxed{\text{must lie in Lie algebra of generalized orthogonal group.}}$

$d\underline{e}' = \underline{e}'\underbrace{(\underline{B}^{-1}d\underline{B})}$

components of linear transformation (pseudo-rotation)
rate of change differential wrt $\underline{e}'$ axes

log takes matrix in group to Lie algebra!

For $\mathbb{R}^3$, rotation group : $\underline{B}^{-1}d\underline{B} = {}^{\shortparallel}\underline{\Omega}' \times {}^{\shortparallel} = \Omega^{i'}\underline{L}_i$ ← basis of Lie algebra $so(3,\mathbb{R})$

↑ primed components of angular velocity differential

under active rotation, linear transformation ~ (1) tensor $\underline{L} \to (\underline{L})$

$\underline{L} \mapsto \underline{B}^{-1}\underline{L}\,\underline{B} = \underline{L}'$ new components

$\underline{B}\,\underline{L}'\underline{B}^{-1} = \underline{L}$ old components

apply to $\underline{B}^{-1}d\underline{B} = \underline{L}' \to \underline{L} = \underline{B}\,\underline{L}'\underline{B}^{-1} = \underline{B}(\underline{B}^{-1}d\underline{B})\underline{B}^{-1}$

$= d\underline{B}\,\underline{B}^{-1} = {}^{\shortparallel}\underline{\Omega}\times{}^{\shortparallel}$

old components of angular velocity differential $= \Omega^{i}\underline{L}_i$

$$\left[\underline{B}^{-1}d\underline{B} = \Omega^{i'}\underline{L}_i\right] \leftrightarrow d\underline{B}\,\underline{B}^{-1} = \Omega^i \underline{L}_i$$

$\underline{B}$ $\quad\underline{B}^{-1}$

$$d\underline{B}\,\underline{B}^{-1} = \Omega^{i'}\underbrace{\underline{B}\,\underline{L}_i\,\underline{B}^{-1}}_{=L_j B^j{}_i} = \Omega^i \underline{L}_i$$

Adjoint action on matrix Lie algebra.

$$(B^j{}_i\,\Omega^{i'})\,\underline{L}_j = \Omega^j \underline{L}_j$$

angular velocity differentials also rotate like components of vector

$$\begin{cases} B^j{}_i\,\Omega^{i'} = \Omega^j \\ \Omega^{i'} = B^{-1\,i}{}_j\,\Omega^j \end{cases}$$

$$(L_i)^j{}_k = \epsilon_{ijk} \quad \underline{\underline{\text{same}}}$$

$$[\underline{L}_i, \underline{L}_j] = \epsilon_{ijk} L_k$$

$[k_i]^j{}_k$ structure constant matrices same as matrix Lie algebra basis.

with orthogonal matrices index position doesnt matter.

$$\epsilon_{ijk} = \epsilon_{mnp} B^m{}_i B^n{}_j B^p{}_j \quad \text{transforms as tensor}$$

$\hookrightarrow$ equivalent with index shuffling to

$$\underline{B}\,\underline{L}_i\,\underline{B}^{-1} = \underline{L}_j B^j{}_i$$

Details in Section 1.7, 4.5 group stuff.

not to worry –

we will work mostly with $SO(3,\mathbb{R})$ rotations

and antisymmetric matrices

$B^{-1}dB$ will be important for "moving frames"

$\boxed{\cancel{A}.3\text{-}4}$

4

$\underline{e}' = [\hat{T}\ \hat{N}\ \hat{B}]$ along curve

$$\text{"}B^{-1}dB\text{"} = \begin{bmatrix} & -\kappa & 0 \\ \kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$$

antisymmetric matrix completely describing curving & twisting of a space curve.

when $\underline{e}' = [\underline{e}_1', \underline{e}_2', \underline{e}_3']$ = ON basis of vector <u>fields</u>

~~this will completely describe the parallel transport.~~

$\boxed{\text{A.3-4.}}$

5

parametrized curve: $\underbrace{\vec{r} = \langle x, y, z\rangle}_{\text{position vector variable}} = \underbrace{\vec{r}(t) = \langle x(t), y(t), z(t)\rangle}_{\text{explicit vector function of parameter } t}$

parametrized surface: $\vec{r} = \vec{r}(u,v)$ now 2 free paramaters

varying them sweeps out a surface

<u>EX</u> graph $z = f(x,y)$

$x = u$
$y = v$
$z = f(u,v)$

$\vec{r} = \langle u, v, f(u,v)\rangle$

$y = f(x)$

set $x = t \to y = f(t)$

$\vec{r} = \langle t, f(t)\rangle$

very restrictive on curves/surfaces (vertical line test)

indices to keep track

general surface

$\vec{r} = \langle x(u^1,u^2), y(u^1,u^2), z(u^1,u^2)\rangle$

(u^1, u^2) <u>ordered</u>

(order matters for orientation)

$\boxed{\text{grid:}}$

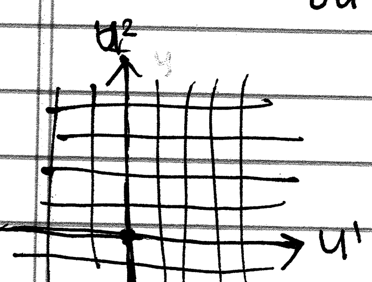
fix equally spaced values of u^2

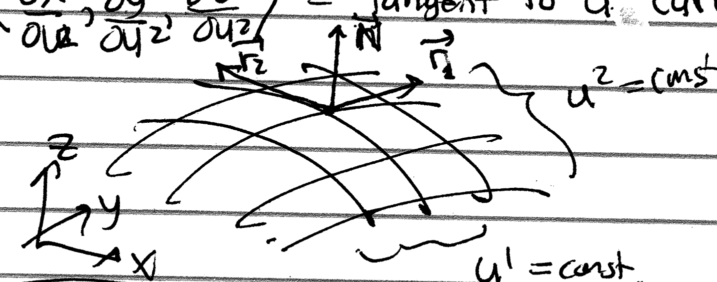
get family of curves with parameter u^1

$\vec{r}_1 \equiv \dfrac{\partial \vec{r}}{\partial u^1} = \left\langle \dfrac{\partial x}{\partial u^1}, \dfrac{\partial y}{\partial u^1}, \dfrac{\partial z}{\partial u^1} \right\rangle$ = tangent to those curves like $\vec{r}\,'(t)$

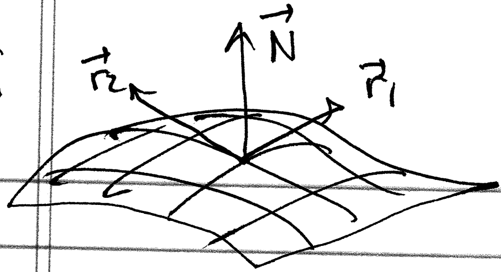
switch u^1, u^2 here:

$\vec{r}_2 \equiv \dfrac{\partial \vec{r}}{\partial u^2} = \left\langle \dfrac{\partial x}{\partial u^2}, \dfrac{\partial y}{\partial u^2}, \dfrac{\partial z}{\partial u^2} \right\rangle$ = tangent to u^2 curves





$\boxed{\vec{N}(u^1,u^2) \equiv \vec{r}_1(u^1,u^2) \times \vec{r}_2(u^1,u^2)}$ normal to tangent plane

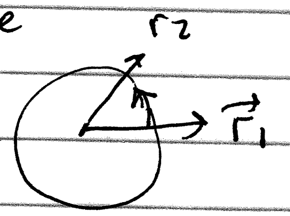


(u^1,u^2) orients surface by choosing a normal direction

right hand rule orients ~~b~~ bases: $\vec{r}_1, \vec{r}_2, \vec{N}$

not ON but obey RH Rule.

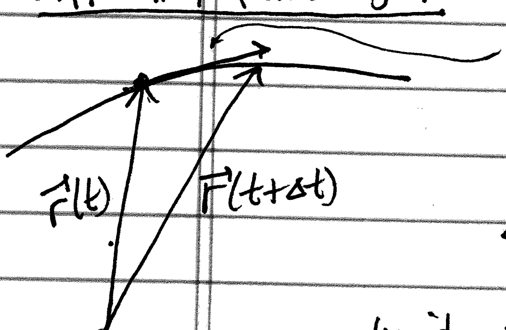
in surface



counterclockwise circles from $\vec{r}_1$ to $\vec{r}_2$ $(<180°!)$

intrinsic orientation to surface like in ordinary plane.

differential of arclength



$$\vec{\Delta r} = \vec{r}(t+\Delta t) - \vec{r}(t) \approx r'(t)\,\Delta t$$

$$\left\{ \begin{array}{l} \vec{r}\,'(t) \approx \dfrac{\vec{\Delta r}}{\Delta t} \\ \vec{\Delta r} \approx \vec{r}\,'(t)\,\Delta t \end{array} \right.$$

$$\Delta s \approx |\Delta \vec{r}| \approx |r'(t)|\Delta t$$

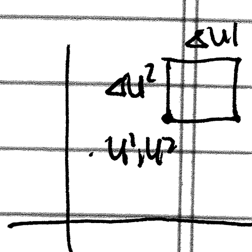
limit $\downarrow$

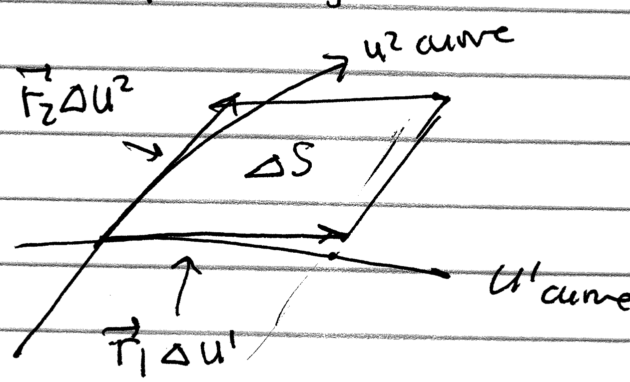
$$ds = |\vec{r}\,'(t)|\,dt$$

$$s = \int ds = \int |\vec{r}\,'(t)|\,dt$$

differential of surface area

repeat along both u^1 & u^2 curves





$$\Delta S = |(\vec{r}_1 \Delta u^1) \times (\vec{r}_2 \Delta u^2)| = \underbrace{|\vec{r}_1 \times \vec{r}_2|}_{|\vec{N}|}\,\Delta u^1 \Delta u^2$$

limit $\downarrow$

$$dS = |\vec{N}|\,du^1 du^2 \rightarrow S = \iint |\vec{N}(u^1,u^2)|\,du^1 du^2$$

A.3-4

7

line integral / surface integral of vector field $\vec{F}(\vec{r})$

$\downarrow$ integrate tangential component wrt ds along curve

$\downarrow$ integrate normal component wrt dS over surface

$\vec{F}(\vec{r}(t))$ along curve

$\vec{F}(\vec{r}(u^1,u^2))$ on surface

$\hat{T}(t)\cdot\vec{F}(\vec{r}(t))$ tangential component. (oriented by t)

$\hat{N}(u^1,u^2)\cdot\vec{F}(\vec{r}(u^1,u^2))$ normal component. (oriented by u^1,u^2)

$$\text{"}\int_C \vec{F}\cdot d\vec{s}\text{"} = \int \vec{F}(\vec{r}(t))\cdot\underbrace{\hat{T}(t)\,|\vec{r}\,'(t)|}_{\vec{r}\,'(t)}\,dt$$

$$= \int \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

$$\text{"}\iint_\Sigma \vec{F}\cdot d\vec{S}\text{"} = \int \vec{F}(\vec{r}(u^1,u^2))\cdot\underbrace{\hat{N}(u^1,u^2)\,|\vec{N}(u^1,u^2)|}_{\vec{N}(u^1,u^2)}\,du^1du^2$$

$$= \underbrace{\int \vec{F}(\vec{r}(u^1,u^2))\cdot\underbrace{\vec{N}(u^1,u^2)}_{\vec{r}_1\times\vec{r}_2}}\,du^1du^2$$

$$|\vec{F}\cdot(\vec{r}_1\times\vec{r}_2)| = |\det(\vec{F},\vec{r}_1,\vec{r}_2)|$$

= volume of parallelopiped.

no sqrts in integrating vector fields along curve / surface even though arclength / surface area has sqrts, difficult integrals.

CHAPTER 11 we will explain in detail

A.3-4

8

Volume integrals in $\mathbb{R}^3$ over parametrized regions

← radius ← polar angle ← azimuthal angle

$r \quad \theta \quad \varphi$

example:

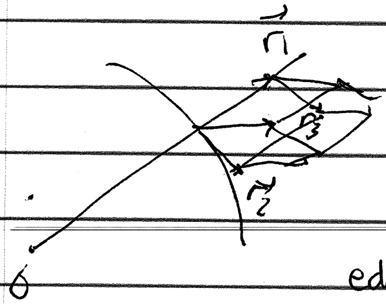
$$\vec r = \vec r(u^1,u^2,u^3) = \langle u^1 \sin u^2 \cos u^3,\ u^1 \sin u^2 \sin u^3,\ u^1 \cos u^2 \rangle$$

spherical coord parametrization of sphere of radius a and interior

$0 \le r \le a,\quad 0 \le \theta \le \pi,\quad 0 \le \varphi \le 2\pi$

$\vec r_i \equiv \dfrac{\partial \vec r}{\partial u^i}$ $\quad (u^1,u^2,u^3)$ ordering orients region

$\vec r_3 \cdot (\vec r_1 \times \vec r_2) > 0$ for right handed frame like spherical coord tan vectors



gridbox at (u^1,u^2,u^3)

Δgridbox for volume integration

edges: $\vec r_1 \Delta u^1,\ \vec r_2 \Delta u^2,\ \vec r_3 \Delta u^3$

volume increment $\Delta V = |\vec r_1 \cdot (\vec r_2 \times \vec r_3)|\, \Delta u^1 \Delta u^2 \Delta u^3$

$\downarrow$

differential $dV = \underbrace{|\vec r_1 \cdot (\vec r_2 \times \vec r_3)|}_{|\det(\vec r_1, \vec r_2, \vec r_3)|}\, du^1 du^2 du^3$ ← abs value not needed in oriented grid

$$V = \int dV = \iiint \vec r_1 \cdot (\vec r_2 \times \vec r_3)\, \underbrace{du^1 du^2 du^3}_{\text{right handed orientation}}$$

inner product (metric) sets length scales along orthogonal directions

determinant extends to general parallelograms, parallelopipeds

$\boxed{4.4}$ $\boxed{4.3}$ 1

$\mathbb{R}^3 : (x, y, z)$ $M^3 : (x, y, t)$ $\underline{G} = \mathrm{diag}(1, 1, \pm 1) = \underline{G}^{-1}$

$$\eta_{ijk} = \sqrt{|\det(\underline{G})|}\, \epsilon_{ijk}$$

$$\eta^{ijk} = G^{im} G^{jn} G^{kp} \eta_{mnp} = \sqrt{|\det(\underline{G})|}\, \underbrace{\epsilon_{mnp} G^{im} G^{jn} G^{kp}}_{\underbrace{\det \underline{G}^{-1}}_{\pm |\det G|^{-1}} \epsilon^{ijk}}$$

$$= \pm \frac{1}{\sqrt{|\det(\underline{G})|}} \epsilon^{ijk}$$

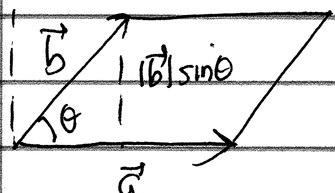
$$\eta_{123} = 1 \longrightarrow \eta^{123} = \pm 1 \qquad G^{33} = \pm 1$$

$$\eta_{ijk} = \epsilon_{ijk} \quad \eta^{ijk} = \pm \epsilon^{ijk}$$

$$\delta^{mn}_{ij} = \delta^m_i \delta^n_j - \delta^n_i \delta^m_j = \epsilon^{mnk} \epsilon_{ijk} \quad \text{identity}$$
$$= \pm \eta^{mnk} \eta_{ijk}.$$

---

2 spacelike vectors: $\vec{a} \cdot \vec{a} > 0$, $\vec{b} \cdot \vec{b} > 0$



$$\text{Area} = |\vec{a}|\,|\vec{b}| \sin\theta$$

$$\text{Area}^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta = |\vec{a}|^2 |\vec{b}|^2 - |\vec{a}|^2 |\vec{b}|^2 \cos^2\theta$$
$$= \vec{a} \cdot \vec{a}\ \vec{b} \cdot \vec{b} - \vec{a} \cdot \vec{b}\ \vec{a} \cdot \vec{b}.$$
$$= \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{b} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{b} \end{vmatrix} = \delta^{mn}_{ij} a^i b^j a_m b_n$$

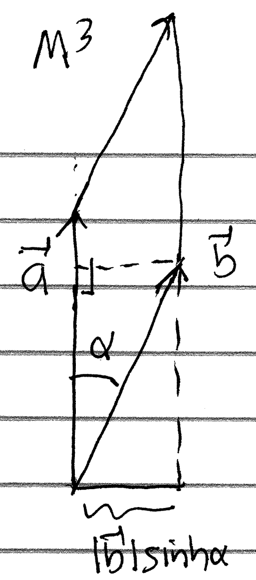
$$= \delta^m_i \delta^n_j a^i b^j a_m b_n - \delta^n_i \delta^m_j a^i b^j a_m b_n$$

$$= \epsilon^{mnk} \epsilon_{ijk} a^i b^j a_m b_n$$
$$= \pm \eta^{mnk} \eta_{ijk} a^i b^j a_m b_n$$
$$= \pm (\eta^{kmn} a_m b_n)(\eta_{kij} a^i b^j)$$
$$= \pm (a \times b)^k (a \times b)_k = \pm (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{b})$$
$$= \pm Q(\vec{a}, \vec{b}, \vec{a}, \vec{b})$$

in M^3 cross prod of 2 spacelike vectors is timelike (3^{rd} direction!) $(a \times b) \cdot (a \times b) < 0$

need minus to get > 0 (Area^2) $Q(\vec{a}, \vec{b}, \vec{c}, \vec{d}) = (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

A.4 | 4.3 | 2



2 timelike vectors: $\vec{a}\cdot\vec{a} < 0$, $\vec{b}\cdot\vec{b} < 0$

$\vec{a}\cdot\vec{b} = -|a||b|\cosh\alpha$

$$\text{Area} = |\vec{a}|(|\vec{b}|\sinh\alpha) \qquad \cosh^2\alpha - \sinh^2\alpha = 1$$

$$\cosh^2\alpha - 1 = \sinh^2\alpha$$

$$\text{Area}^2 = |\vec{a}|^2|\vec{b}|^2\cosh^2\alpha - |\vec{a}|^2|\vec{b}|^2\sinh^2\alpha$$

$$= (\vec{a}\cdot\vec{b})^2 - (a\cdot a)(\vec{b}\cdot\vec{b})$$

$$= -\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} \end{vmatrix} = -\delta^{mn}_{ij}\, a^i b^j a_m b_n$$

$$= \eta_{kij}\,\eta^{kmn}\, a^i b^j a_m b_n$$

$$= (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b}) > 0$$

$$= Q(a,b,a,b)$$

cross-product of 2 timelike vectors is spacelike.

where $Q(\vec{a},\vec{b},\vec{c},\vec{d}) = (\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d})$

quadruple scalar product

antisymmetric matrices: $\underline{A}$ $\underline{B}$

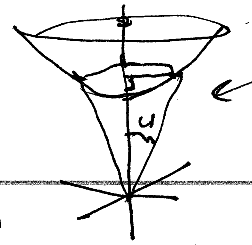
$= A^{ij}B_{ij}$ usual inner product for vector space of asym matrices!!

~~index raising/lowering~~

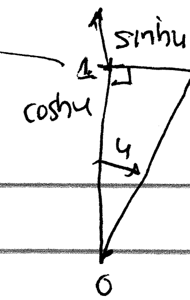
conventions extend to M^3 from E^3.

$\boxed{4.4}$ 3
$\boxed{4.3}$

surfaces



unit pseudosphere



$\mathbb{R}^3: \quad x^2+y^2-z^2=-1$ $\qquad$ $M^3: \quad x^2+y^2-t^2=-1$

$x = \sinh u \cos v$ $\qquad$ $x = \sinh u \cos v$
$y = \sinh u \sin v$ $\qquad$ $y = \sinh u \sin v$
$z = \cosh u$ $\qquad$ $t = \cosh u$

$$\vec{r} = \langle \sinh u \cos v, \sinh u \sin v, \cosh u \rangle$$

$$\boxed{\begin{array}{l} u = 0 \,..\, \operatorname{arccosh} 2 \\ z = t = 1 \,..\, 2 \end{array}}$$

polar $\quad \vec{r}_1 = \langle \cosh u \cos v, \cosh u \sin v, \sinh u \rangle$

azimuthal $\quad \vec{r}_2 = \langle -\sinh u \sin v, \sinh u \cos v, 0 \rangle$

$$\vec{N}_E = \vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} i & j & k \\ cC & cS & s \\ -sS & sC & 0 \end{vmatrix} = \langle 0 - s^2C, -s^2S - 0, csC^2 + csS^2 \rangle$$

$$= \sinh u \langle \cancel{\cos v, -s} -\sinh u \cos v, -\sinh u \sin v, \cosh u \rangle$$

$$\vec{N}_M = \sinh u \langle -\sinh u \cos v, -\sinh u \sin v, -\cosh u \rangle$$

raise index on covector

$$N_E \bullet N_E = \sinh^2 u \left[\sinh^2 u (\cos^2 v + \sin^2 v) + \cosh^2 u \right]$$

$$\vec{N}_M \circ \vec{N}_M = \sinh^2 u \left[\underbrace{\sinh^2 u (\cos^2 v + \sin^2 v) - \cosh^2 u}_{-1} \right]$$

$$|\vec{N}_M| = \sqrt{-\vec{N}_M \circ \vec{N}_M} = \sinh u \longrightarrow dS_M = \sinh u \, du \, dv$$

$$\updownarrow$$

$$\boxed{\begin{array}{l} dS = \sin\theta \, d\theta \, d\varphi \\ \text{on unit sphere} \end{array}}$$

$$|\vec{N}_E| = \sqrt{\vec{N}_E \bullet \vec{N}_E} = \sinh u \sqrt{\cosh^2 u + \sinh^2 u}$$

$$dS_E = \sinh u \sqrt{\cosh^2 u + \sinh^2 u} \, du \, dv$$

$$S_M = \int_0^{2\pi} \int_0^{\operatorname{arccosh} 2} \sinh u \, du \, dv = 2\pi \cosh u \Big|_0^{\operatorname{arccosh} 2}$$

$$= 2\pi \left[\cosh(\operatorname{arccosh}(2)) - \cosh(0) \right] = 2\pi(2-1) = 2\pi$$

$$\approx 6.28$$

$$S_E = \int_0^{2\pi} \int_0^{\operatorname{arccosh} 2} \sinh u \sqrt{\cosh^2 u + \sinh^2 u} \, du \, dv$$

Maple

$$= 2\pi \int_0^{\operatorname{arccosh} 2} \sinh u \sqrt{\cosh^2 u + \sinh^2 u} \, du \approx 11.66$$