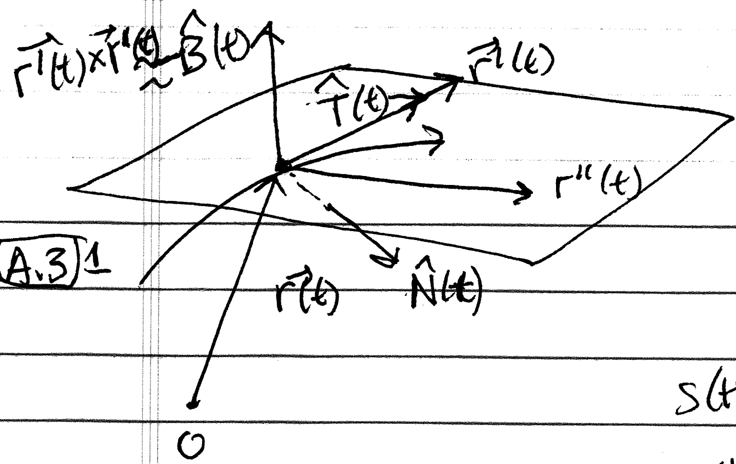


$\boxed{A.3}$ 1

osculating plane = vel-acc plane

$$\frac{ds}{dt} = |\vec r'(t)| = \text{speed}$$

$$ds = |r'(t)|\, dt$$

$$s(t) = \int ds = \int |\vec r'(t)|\, dt$$

"invert" $t = t(s)$

$\vec r(t) \to \vec r(s)$ arclength parametrized

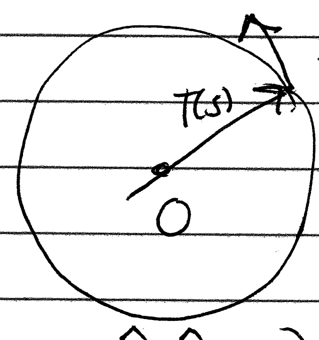
$\boxed{\text{VERY SLOPPY Valentine's Day}}$

$$\frac{d\vec r(s)}{ds} = \frac{dr/dt}{ds/dt} = \frac{r'(t)}{\|r'(t)\|} = \hat T(t)$$

unit speed parametrization

only geometry relevant not how trace out curve.

$$\frac{d^2\vec r(s)}{ds^2} = \frac{d\hat T(s)}{ds} = \underbrace{K(s)}_{\substack{\text{magnitude } \geq 0\\ \text{"curvature"}}}\underbrace{\hat N(s)}_{\text{direction}}$$



$\frac{d\hat T}{ds} \perp \hat T$ can only rotate.

$\hat T, \hat N, \hat B$

1 2 3 right handed ON basis.

$\hat B = \widehat{r' \times r''}$

$= \hat T \times \hat N$ etc.

$$\frac{d}{ds}\left(\hat T \cdot \hat T = 1\right)$$

$$\frac{dT}{ds}\cdot T + T\cdot\frac{dT}{ds} = 0$$

$$\hookrightarrow 2\hat T \cdot \frac{d\hat T}{ds} = 0$$

$e_{a'} = e_b B^a{}_a(t) \leftarrow$ rotation to new basis $(\hat T, \hat N, \hat B) \equiv (e_{a'})$

$e' = eB$

$e' = eB' = eBB^{-1}B'$

$= e'\underbrace{B^{-1}B'}_{\text{antisymmetric}}$ (Lie algebra of rotation group)

$$\left[\frac{dT}{ds}, \frac{dN}{ds}, \frac{dB}{ds}\right] = [T\ N\ B]\underbrace{\omega}_{\text{antisymmetric matrix}}$$

antisymmetric

$$= [T\ N\ B]\begin{bmatrix} 0 & -K & \\ K & 0 & \\ 0 & \tau & \end{bmatrix}$$

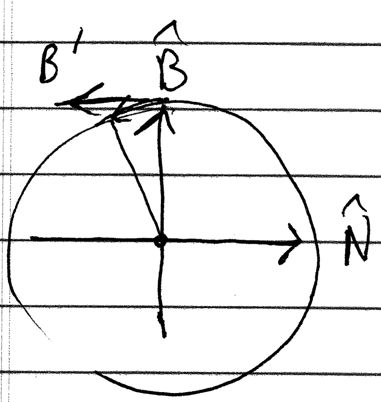
$\boxed{A.3}$ 2

$$\frac{dN}{dS} = -K\hat{T} + \underset{\substack{\parallel \\ \tau}}{?}\,\hat{B} \leftarrow (\text{must be} \perp B!)$$

"torsion"

$$\left[\frac{dT}{ds}\ \frac{dN}{ds}\ \frac{dB}{ds}\right] = [T\ N\ B]\begin{bmatrix} 0 & -k & 0 \\ k & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix}$$

$$\frac{dB}{ds} = -\tau\hat{N}$$

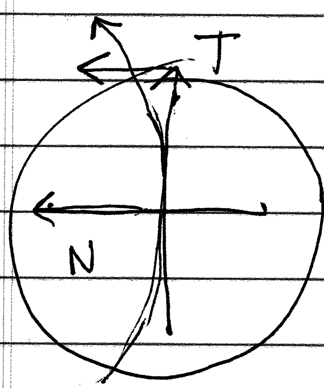


normal to osc plane rotates if $\tau \neq 0$

$\leftrightarrow$ nonplanar curve

$\tau = 0$, $\hat{B}$ fixed motion takes place in fixed plane.

$$\frac{dT}{ds} = k\hat{N}$$



$\boxed{A.3}$ 3

circle:

$\vec{r} = a\langle \cos t, \sin t\rangle \quad = a(\cos t\,\hat{i} + \sin t\,\hat{j}) + \vec{O}$ ← center

$\vec{r}\,' = a\langle -\sin t, \cos t\rangle$

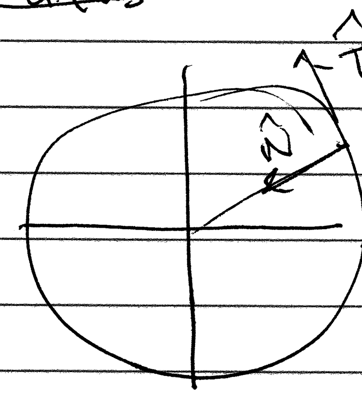
$|\vec{r}\,'| = a = \frac{ds}{dt} \rightarrow s = \int a\,dt = at + \cancel{C}^{0}$

$t = \frac{s}{a} \qquad \frac{dt}{ds} = \frac{1}{a} \qquad \frac{d\theta}{ds} = \frac{1}{a} = K$

$\vec{r} = a\langle \cos\frac{s}{a}, \sin\frac{s}{a}\rangle$ ~~$= a(\cos$~~

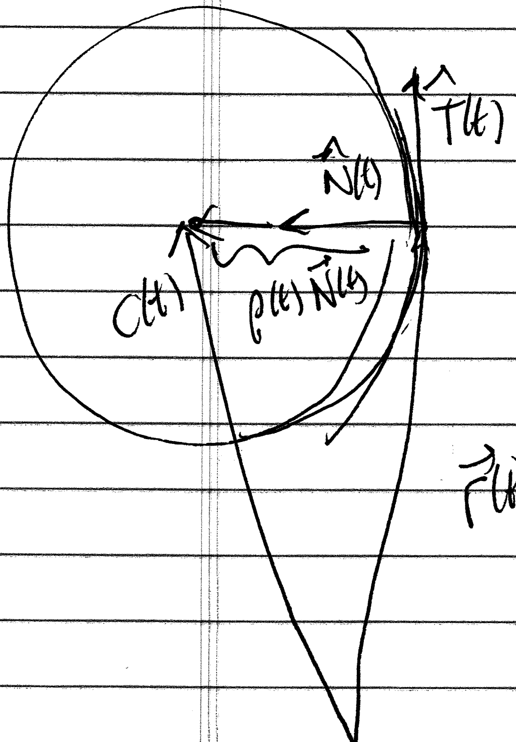
$\frac{dr}{ds} = \langle -\sin\frac{s}{a}, +\cos\frac{s}{a}\rangle = \hat{T}$

$\frac{d^2r}{ds^2} = \frac{d\hat{T}}{ds} = \underbrace{\frac{1}{a}}_{K}\underbrace{\langle -\cos\frac{s}{a}, -\sin\frac{s}{a}\rangle}_{\hat{N}}$



$K = \frac{1}{a}$ for circle.

$K(s) \equiv 1/\rho(s) \rightarrow \rho(s)$ radius of curvature for equivalent circle



$\vec{r} = \vec{O} + a(\cos t\,\hat{i} + \sin t\,\hat{j})$

↓

$\vec{r}(t,\theta) = C(t) + \rho(t)(\cos\theta(-\hat{N}) + \sin\theta\,\hat{T})$

$\vec{r}(t) + \rho(t)\hat{N}(t)$ cancel $\theta = 0$

osc circle at $\vec{r}(t)$

circle of best fit

$\vec{r}(t, 0) = \vec{r}(t)$ pt of tangency.

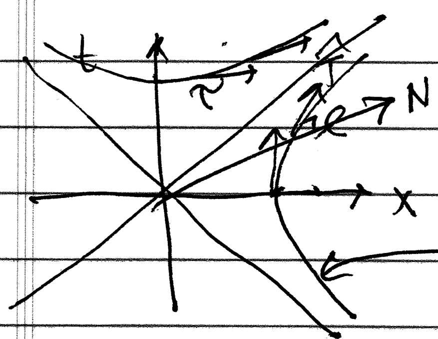
$\boxed{A.3}$ 4. Everything we did can be repeated for any inner product → spacetime

$$\underline{G} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ or submatrix $n = 2, 3, 4$ dimension.

<u>$n=2$ Lorentz plane.</u>

$\pm s^2 = -t^2 + x^2$ $> 0 \to \ell^2$ proper distance s

$< 0 \to -\tau^2$ proper time τ.



pseudo-circles. centered at origin

timelike curve tangent "timelike"

$\vec{r} = \ell \langle \sinh\alpha, \cosh\alpha \rangle$ ⟵ $\vec{r}(0) = \langle 0, \ell \rangle$

$\qquad\qquad t \qquad x$

$\vec{r}\,' = \ell \langle \cosh\alpha, \sinh\alpha \rangle$

$-|\vec{r}\,'|^2 = \ell^2(-\cosh^2\alpha + \sinh^2\alpha) = -\ell^2$ $\qquad |r'| = \ell = \frac{d\tau}{d\alpha}$

$\int d\tau = \int \ell \, d\alpha$

$\hat{T} = \langle \cosh\frac{\tau}{\ell}, \sinh\frac{\tau}{\ell} \rangle$ $\qquad \tau = \ell\alpha$

$\frac{d\hat{T}}{d\tau} = \underbrace{\frac{1}{\ell}}_{K} \underbrace{\langle \sinh\frac{\tau}{\ell}, \cosh\frac{\tau}{\ell} \rangle}_{\hat{N}}$ $\qquad \alpha = \frac{\tau}{\ell}$

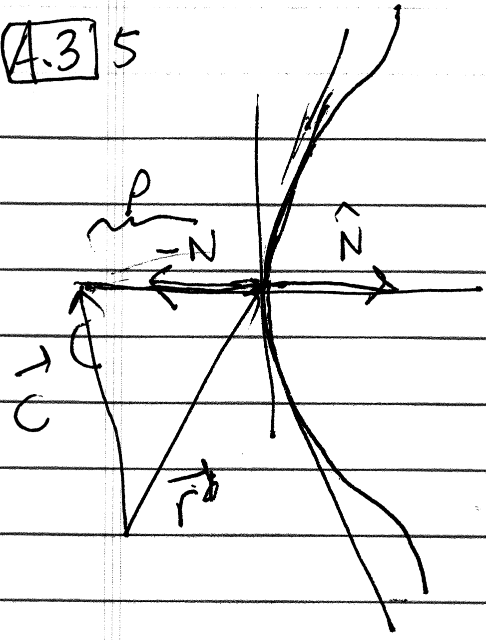
$(-s^2 + c^2) = 1$ spacelike unitvector

$K = \frac{1}{\ell}$

pseudoradius $= 1/K$. but center at origin is distance ℓ along $-\hat{N}$.

$r = \vec{0} + \ell(\sinh\alpha\, \hat{j} + \cosh\alpha\, \hat{i})$

$\boxed{4.3}$ 5



$$\cancel{\vec{C}(t) = \vec{r}(t) + \rho(-\hat{N}(t))}$$

$$\vec{r} = \vec{O} + \ell(\sinh\beta\,\hat{j} + \cosh\beta\,\hat{i})$$ pseudo circle

$$\begin{array}{cccc} \vec{O} & \ell & \hat{j} & \hat{i} \\ \downarrow & \downarrow & \downarrow & \downarrow \\ \vec{C} & \rho & \hat{T} & \hat{N} \end{array}$$

$$\vec{r}(\tau,\beta) = \vec{C}(\tau) + \rho(\tau)\left(\sinh\beta\,\hat{T} + \cosh\beta\,\hat{N}\right)$$

$$\vec{r}(\tau,0) = \vec{r}(t) + \rho(-N(t)) + \rho(t)N(t)$$

$$= \vec{r}(t)$$