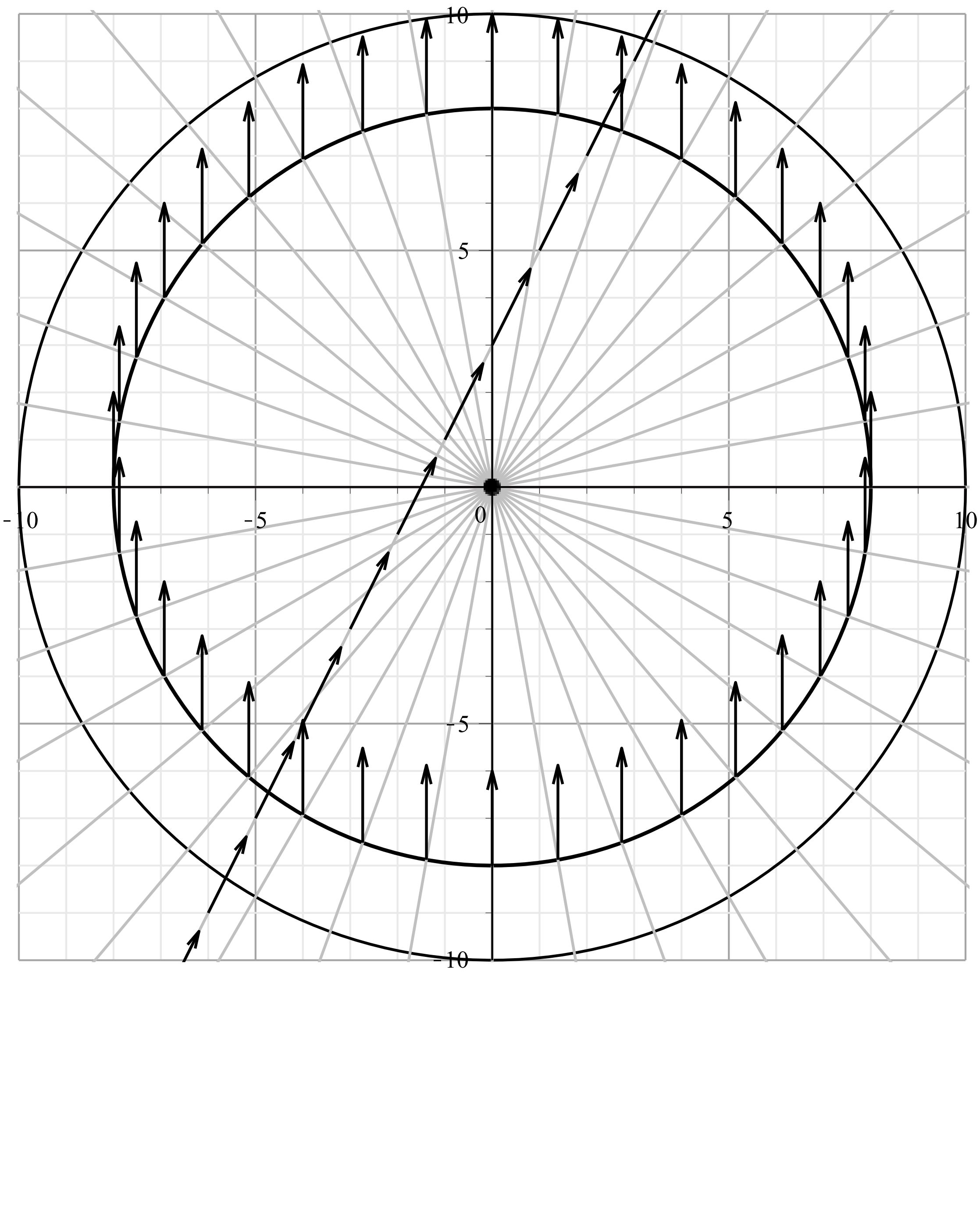
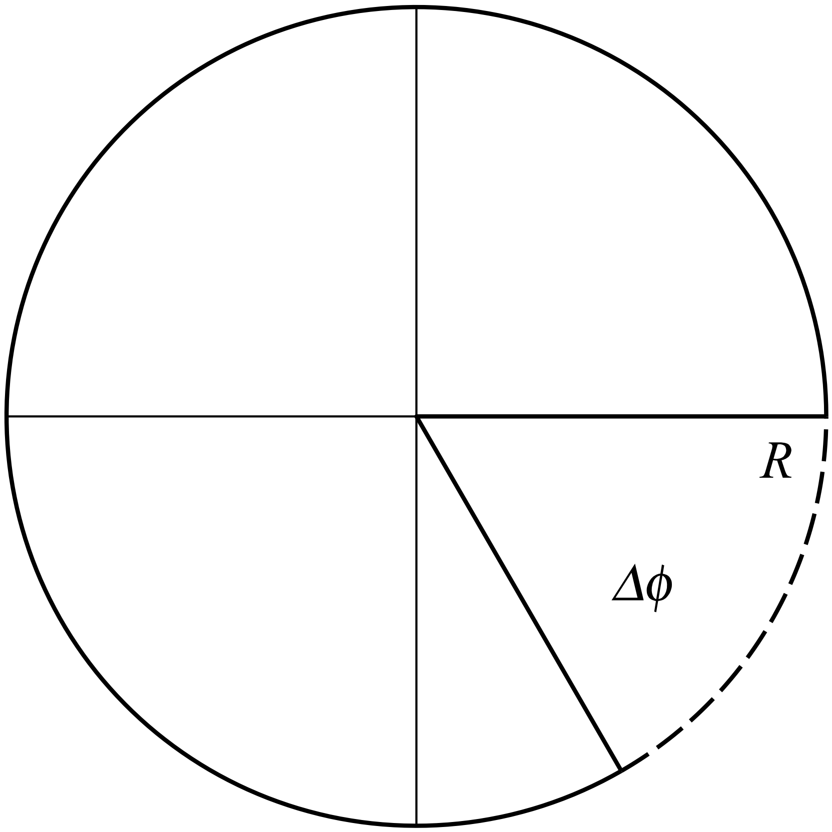
the polar coordinate grid
and
parallel transport on a cone

- **polar coordinate grid plots**

- **the cone maker**


10
5
-5
-10
-10
-5
0
5
10

Cut this along the positive x-axis, then fold the lower 4th quadrant sector underneath that half ray, lining it up with one of the 10 degree rays from about $\Phi = \frac{3\,\pi}{2} + \frac{\pi}{6}$ (30 degrees counterclockwise past the negative y-axis) to about $\Phi = \frac{7\,\pi}{6}$ (30 degrees counterclockwise past the negative x-axis) to make a cone. You can slide this along that range of angles to see how things change, indeed you can start from zero overlap and increase through this range. [The paper seems to have overlap problems as you approach π.]



1) Notice how much the "constant" vertical vector of length 2 along the circle of radius 8 is rotated with respect to its initial value on the positive x-axis if we trace out the circle in the counterclockwise direction and then compare with its value at the chosen overlap angle Φ once folded into a cone by removing a sector of the circle through the folding operation. Let $\Delta\phi = 2\,\pi - \Phi$ be this "defect angle" of the cone: how much less than $2\,\pi$ of the original circle is represented in the new circle on the cone with this angle removed. This is the angle of the sector which is slid under the positive x-axis. Notice that the final value is rotated in the counterclockwise direction relative to the initial value, with an angle that increases with the defect angle. Can you guess how these two are related? Look at the case $\Delta\phi = \frac{\pi}{3}$ (60 degrees). Can you prove your guess is right?

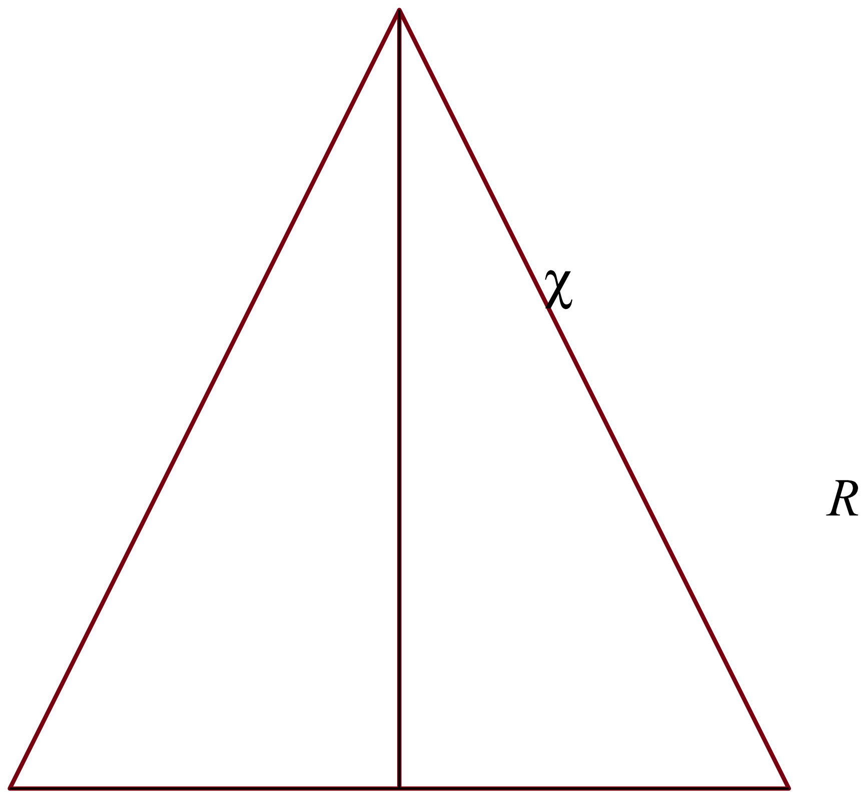
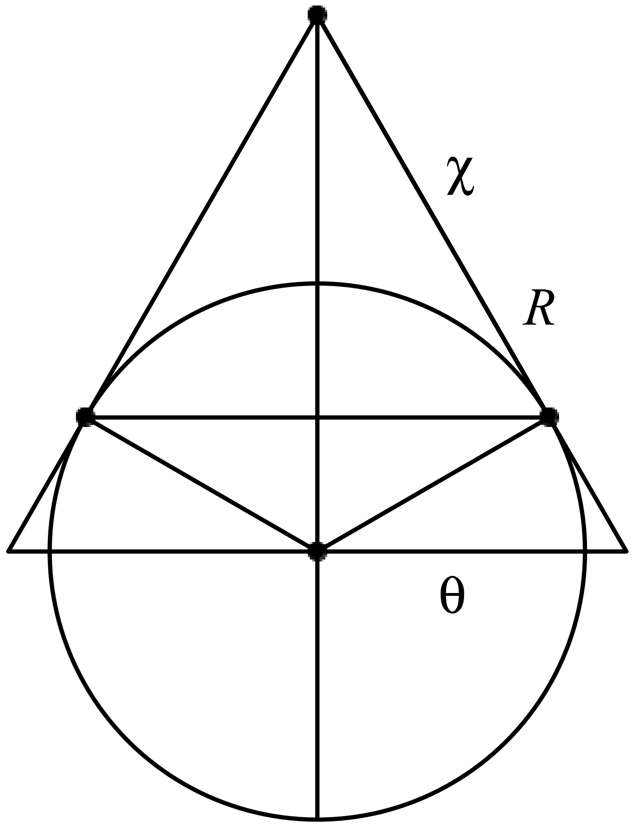


Figure out how this angle $\Delta\phi$ is related to the half-opening angle χ of the cone, namely the angle of opening of the cone from its symmetry axis. This requires drawing a cross-sectional diagram through the polar axis and using high school trigonometry of the right triangle formed in this cross-section by the solid cone of lateral side length R, the radius of the original circle in the plane. [Hint: compare the circumference of the cone base with circumference of the original circle of the same radius.]

2) Look at the straight line $y - 1 = 2\ (x + 1)$ shown with the same length arrows lined up along its path, but in the conical configuration. Notice that as you rotate the cone, this path seems to keep its direction straight withing the surface. It is autoparallel. Its tangent vector does not rotate left or right as we move along the "extrinsically" curved surface but always points directly forward along the path. This defines a so called geodesic. Of course the geodesics of the cone are the straight lines of the corresponding plane which governs the "intrinsic geometry of the cone: flat just like the plane. Of course as a vector in space, it is changing direction, but only rotating in space to remain tangent to the surface without any extra rotation left or right within the surface. This same feature characterizes the great circles on the sphere, as anyone knows who has taken a trans-atlantic flight and watched the flight tracker arch the path up and then down to get to Europe, following a great circle. Obvious when you look at a globe instead of a flat map. The sphere has a truly curved intrinsic geometry. There is no global parallelism like on the plane or the cone. We will learn how to quantify the intrinsic and extrinsic geometry of surfaces in space (and spacetime!).



3) Now consider a tangent cone to a sphere at a circle of constant polar angle θ measured down from the north pole. Express ithe cone's half opening angle χ in terms of the polar angle. This again requires drawing a cross-sectional diagram through the polar axis. If the sphere were the Earth, how is χ related to the latitude of the circle? If we express $\Delta\phi$ from part 1) in terms of χ and then in terms of θ, we get the amount by which a vector is rotated by "parallel transporting" it around a polar circle of polar angle θ on a sphere of radius $a = \dfrac{R}{\tan\chi}$.
We will calculate this later by solving a differential equation.

► more figures