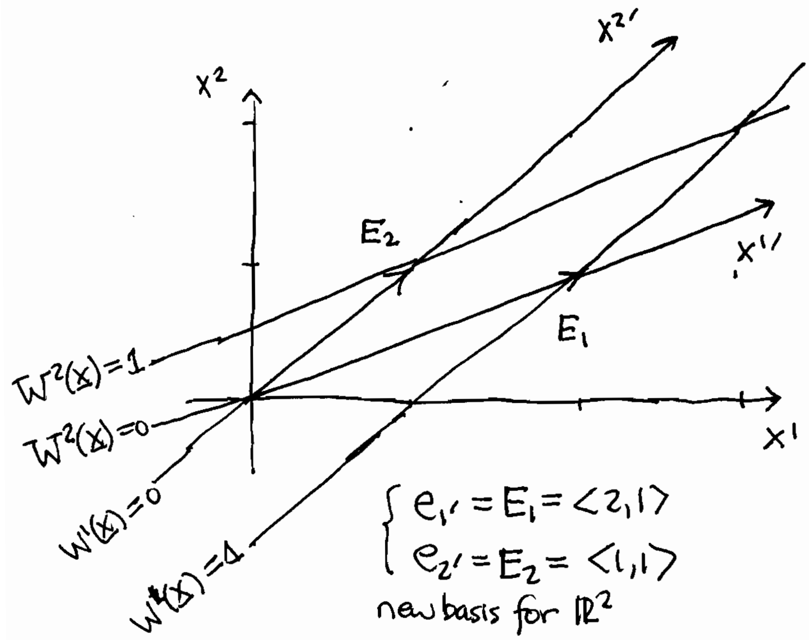

1) Extend E_1, E_2 to new coordinate axes.
2) Complete them to a parallelogram.
3) Extend the sides of the parallelogram
Each pair of opposite sides represents graphically to two dual basis covectors but in a complementary way: the lines of W^1 are parallel to E_2 and the lines of W^2 are parallel to E_1.
4) Tile the plane with the basic unit parallelogram to get the new unit coordinate grid whose vertices (intersections of grid lines) correspond to new integer coordinate points.

$$\begin{cases} e_{1'} = E_1 = \langle 2,1\rangle \\ e_{2'} = E_2 = \langle 1,1\rangle \end{cases}$$
new basis for $\mathbb{R}^2$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$W^i = A^i{}_j\,\omega^j \qquad \begin{bmatrix} W^1 \\ W^2 \end{bmatrix} = \underline{A}\begin{bmatrix} \omega^1 \\ \omega^2 \end{bmatrix}$$

$$\underline{B} = \langle \underline{E}_1 | \underline{E}_2 \rangle \qquad E_i = B^j{}_i e_j = e_j B^j{}_i$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad [E_1 E_2] = [e_1, e_2]\,\underline{B}$$

$$\left.\begin{aligned} \omega^{1'} &= W^1 = a\omega^1 + b\omega^2 = ax^1 + bx^2 \\ \omega^{2'} &= W^2 = c\omega^1 + d\omega^2 = cx^1 + dx^2 \end{aligned}\right\}$$ just 2 linear functions on plane $\rightarrow$

duality conditions:

row 1

$$1 = W^1(E_1) = (a\omega^1 + b\omega^2)(\langle 2,1\rangle) = a\omega^1(\langle 2,1\rangle) + b\omega^2(\langle 2,1\rangle) = a(2) + b(1)$$
$$0 = W^1(E_2) = (a\omega^1 + b\omega^2)(\langle 1,1\rangle) = a\omega^1(\langle 1,1\rangle) + b\omega^2(\langle 1,1\rangle) = a(1) + b(1)$$

row 2

$$0 = W^2(E_1) = (c\omega^1 + d\omega^2)(\langle 2,1\rangle) = c\omega^1(\langle 2,1\rangle) + d\omega^2(\langle 2,1\rangle) = c(2) + d(1)$$
$$1 = W^2(E_2) = (c\omega^1 + d\omega^2)(\langle 1,1\rangle) = c\omega^1(\langle 1,1\rangle) + d\omega^2(\langle 1,1\rangle) = c(1) + d(1)$$

so $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\rightarrow$ solve with $(\underline{B}^T)^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

same coeff matrix $\underline{B}^T$
(rows are components of E_1, E_2)

$\begin{bmatrix} a \\ b \end{bmatrix} = B^{T-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} c \\ d \end{bmatrix} = B^{T-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} a & c \\ b & d \end{bmatrix}}_{\underline{A}^T} = \underline{B}^{T-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{B}^{T-1}$

same coeff matrix $\underline{B}^{T-1}$

$$\underline{A} = ((\underline{B}^T)^{-1})^T = \underline{B}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

(note $(\underline{B}^{-1})^T = (\underline{B}^T)^{-1}$, $(\underline{B}^T)^T = \underline{B}$)

so $W^1 = \omega^1 - \omega^2 = x^1 - x^2$
$W^2 = -\omega^1 + 2\omega^2 = -x^1 + 2x^2$

$$W^2(\langle 2,-1\rangle) = (-\omega^1 + 2\omega^2)(\langle 2,-1\rangle) = -(2) + 2(-1) = -4$$ $\leftarrow$ go to $-4E_2$ on $x^{2'}$ axis
$$W^1(\langle 2,-1\rangle) = (\omega^1 - \omega^2)(\langle 2,-1\rangle) = (2) - (-1) = 3$$ $\leftarrow$ go to $3E_1$ on $x^{1'}$ axis

} complete parallelogram — diagonal is $\langle 2,-1\rangle$

In matrix notation $x^{i'} = W^i(\underline{x}) = A^i{}_j\,\omega^j(\underline{x}) = A^i{}_j\,x^j$

$$x^i = \underbrace{(A^{-1})^i{}_j\,W^j}_{\omega^i}(\underline{x}) = B^i{}_j\,x^{j'}$$

$$\begin{bmatrix} x^{1'} \\ x^{2'} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-(-1) \\ -(2)+2(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 2(3)+1(-4) \\ 1(3)+1(-4) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$