Why LinAlg with DE handout revisited   PREVIEW!!

$$\left.\begin{aligned}\frac{dx_1}{dt} &= x_2 = 0x_1 + 1x_2 \\ \frac{dx_2}{dt} &= x_1 = 1x_1 + 0x_2\end{aligned}\right\} \rightarrow \underline{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

matrix form: $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\frac{d}{dt}\vec{x} = A\vec{x}$ or $\boxed{\vec{x}' = A\vec{x}}$

eigenvectors: $A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$ $0 = \det(A - \lambda I) = \lambda^2 - 1 = (\lambda-1)(\lambda+1) \rightarrow \lambda = 1, -1$

$\lambda = 1$: $A - 1I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{rref} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = t$, $x_2 = t$ $\underline{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{b}_1}$

$\lambda = -1$: $A + 1I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{rref} \begin{array}{cc} L & F \\ \end{array} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_1 = -t$, $x_2 = t$ $\underline{x} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{b}_2}$

$\{\vec{b}_1, \vec{b}_2\}$ new basis of plane

$\underline{B} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\underline{B}^{-1}\underline{A}\,\underline{B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \equiv A_B$ diagonalized matrix of eigenvalues.

change variables

$\vec{x} = B\vec{y}$, $\vec{y} = B^{-1}\vec{x} \longrightarrow B^{-1}[\vec{x}' = A\vec{x}] \rightarrow (B^{-1}\vec{x})' = B^{-1}A\vec{x} = B^{-1}A(B\vec{y})$

$\vec{y}' = (B^{-1}AB)\vec{y}$

$\boxed{\vec{y}' = A_B\vec{y}}$

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix}$

uncoupled eqns

$y_1' = y_1 \rightarrow y_1 = c_1 e^{t}$

$y_2' = -y_2 \longrightarrow y_2 = c_2 e^{-t}$

eigenvalues are exponential rate factors

$[\, y' = ky \rightarrow y = ce^{kt} \,]$

transform back

eigenvalue, eigenvector

$\vec{x} = B\vec{y} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 e^{t} \\ c_2 e^{-t} \end{bmatrix} = \underbrace{c_1 e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\text{growing mode}} + \underbrace{c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{decaying mode}}$

↓ do multiplication

$= \begin{bmatrix} c_1 e^{t} - c_2 e^{-t} \\ c_1 e^{t} + c_2 e^{-t} \end{bmatrix} \Leftrightarrow \boxed{\begin{aligned} x_1 &= c_1 e^{t} - c_2 e^{-t} \\ x_2 &= c_1 e^{t} + c_2 e^{-t} \end{aligned}}$ general soln in scalar form

initial conditions: $x_1(0) = 1$, $x_2(0) = 0$. $\rightarrow e^{0t} = 1$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \xrightarrow{\text{solve}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$

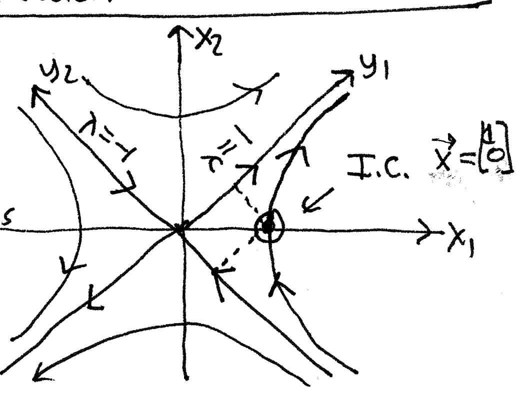
$\vec{x}(0) = B\vec{y}(0) \rightarrow \vec{y}(0) = B^{-1}\vec{x}(0)$

> solving for I.C.s is just changing the coords of the initial vector.

$\vec{x} = \frac{1}{2}e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ vector form (showing 2 modes)

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{t} + \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} \end{bmatrix}$ scalar form (almost)

pure exp growth/decay along eigendirections



Visualizing eigendirections with a phaseplot

> *with*(*DEtools*) : *with*(*LinearAlgebra*) :

The 2x2 matrix of coefficients **A** of the rhs of the vector differential equation **x ' = A x is:**

> $A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \langle x1'(t), x2'(t)\rangle = A.\langle x1(t), x2(t)\rangle$

Note: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$A := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathrm{D}(x1)(t) \\ \mathrm{D}(x2)(t) \end{bmatrix} = \begin{bmatrix} x2(t) \\ x1(t) \end{bmatrix} \tag{1}$$

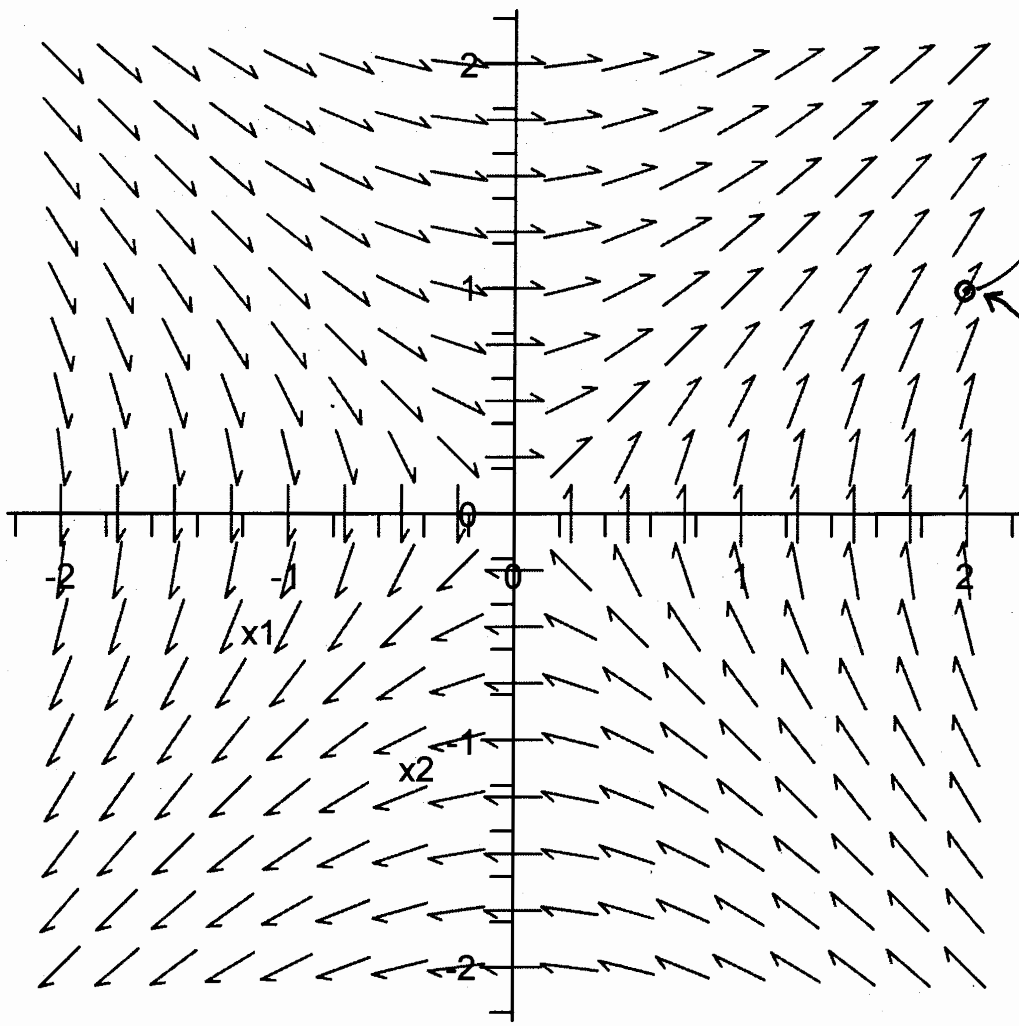
For dsolve and DEplot, we need the system of scalar equations:

> $deqs := x1'(t) = x2(t), x2'(t) = x1(t)$

$$deqs := \mathrm{D}(x1)(t) = x2(t), \mathrm{D}(x2)(t) = x1(t) \tag{2}$$

We can then look at the phaseplot of this differential equation in the x1-x2 plane, like the directionfield tangent line plot for a general first order equation in the *t-x* plane, but now with an arrowhead to indicate the direction as well as the slope.
[The dirgrid option enables us to change the default 20x20 gridpoint field of arrows if we wish.]

> $DEplot([deqs], [x1(t), x2(t)], t = 0..1, x1 = -2..2, x2 = -2..2, dirgrid = [17, 17])$



example point:

pt [2,1]
arrow is scaled down vector with direction $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

slope 2

17 x 17 grid equally divides window in 16 x 16 equal squares of width 2/8 = 1/4

Notice that there are two solution curves (which connect up the arrows) where the arrows line up along a straight line through the origin (linear subspace). One can measure their approximate slopes by clicking on the