PREVIEW: Why LinAlg with DE?

$$\boxed{\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= x\end{aligned}}$$

coupled linear 1st order DEs

($x, y, \frac{dx}{dt}, \frac{dy}{dt}$ appear only linearly, can't solve one DE without solving other)

↓ LinAlg

$$+: \quad \frac{dx}{dt} + \frac{dy}{dt} = y + x \quad \rightarrow \quad \frac{d}{dt}\overbrace{(x+y)}^{u} = \overbrace{(x+y)}^{u}$$

$$-: \quad \frac{dx}{dt} - \frac{dy}{dt} = y - x \quad \rightarrow \quad \frac{d}{dt}\underbrace{(x-y)}_{v} = -\underbrace{(x-y)}_{v}$$

uncoupled linear DEs

$$\rightarrow \boxed{\begin{aligned}\frac{du}{dt} &= u\\ \frac{dv}{dt} &= -v\end{aligned}} \quad \xrightarrow{\text{solve}} \quad \begin{aligned} u &= c_1 e^{t}\\ v &= c_2 e^{-t}\end{aligned}$$

$$\downarrow$$

$$\boxed{\begin{aligned} u &= x+y\\ v &= x-y\end{aligned}}$$

linear change of variables

↓ LinAlg

$$+: \quad u+v = 2x$$
$$-: \quad u-v = 2y$$

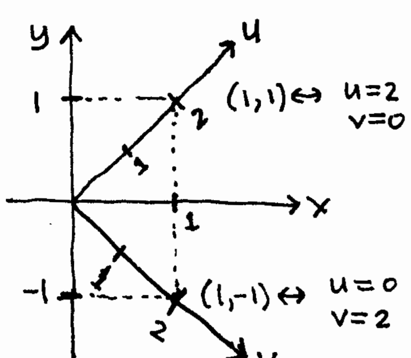
$$\rightarrow \boxed{\begin{aligned} x &= \tfrac{1}{2}(u+v)\\ y &= \tfrac{1}{2}(u-v)\end{aligned}}$$

inverse transformation

back substitute

geometric interpretation:

space of unknowns: x-y plane



linear change of coordinates in x-y plane

general solution

$$\boxed{\begin{aligned} x &= \tfrac{1}{2}(c_1 e^{t} + c_2 e^{-t})\\ y &= \tfrac{1}{2}(c_1 e^{t} - c_2 e^{-t})\end{aligned}}$$

I.C.s ↓

$$\begin{aligned} x(0) &= \boxed{\tfrac{1}{2}(c_1 + c_2) = x_0}\\ y(0) &= \boxed{\tfrac{1}{2}(c_1 - c_2) = y_0}\end{aligned}$$

unknowns given

2x2 linear system of eqns

↓ LinAlg

$$\begin{aligned} +: &\quad \boxed{c_1 = x_0 + y_0}\\ -: &\quad \boxed{c_2 = x_0 - y_0}\end{aligned}$$

soln

back substitute

IVP soln:

$$x = \tfrac{1}{2}\left[(x_0+y_0)e^{t} + (x_0-y_0)e^{-t}\right] = x_0\left(\frac{e^{t}+e^{-t}}{2}\right) + y_0\left(\frac{e^{t}-e^{-t}}{2}\right)$$

$$y = \tfrac{1}{2}\left[(x_0+y_0)e^{t} - (x_0-y_0)e^{-t}\right] = x_0\underbrace{\left(\frac{e^{t}-e^{-t}}{2}\right)}_{\sinh t} + y_0\underbrace{\left(\frac{e^{t}+e^{-t}}{2}\right)}_{\cosh t}$$

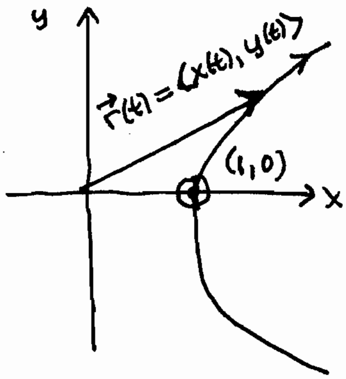
regroup x_0, y_0 terms

linear combinations of exponentials

LinAlg operations:
addition, subtraction, multiplication by a constant

PREVIEW: Why Lin Alg with DE? (2)

$\boxed{\text{interpretation}}$



Suppose we consider the initial data point $(x_0,y_0)=(1,0)$ in the x-y plane.

Then the solution represents a parametrized curve in the x-y plane whose position vector is

$$\vec{r}(t)=\langle x(t),y(t)\rangle = \langle \cosh t, \sinh t\rangle = \left\langle \frac{e^t+e^{-t}}{2}, \frac{e^t-e^{-t}}{2}\right\rangle$$

and tangent vector is

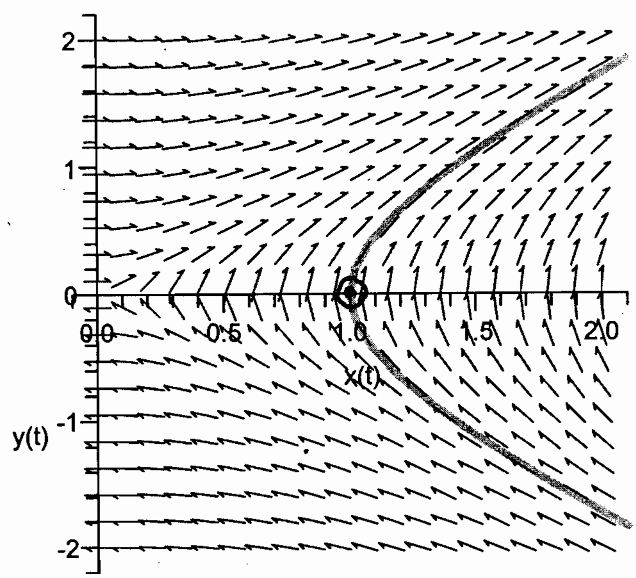
$$\vec{r}\,'(t)=\langle x'(t),y'(t)\rangle = \langle \sinh t, \cosh t\rangle = \left\langle \frac{e^t-e^{-t}}{2}, \frac{e^t+e^{-t}}{2}\right\rangle$$

in vector form the DE system is just

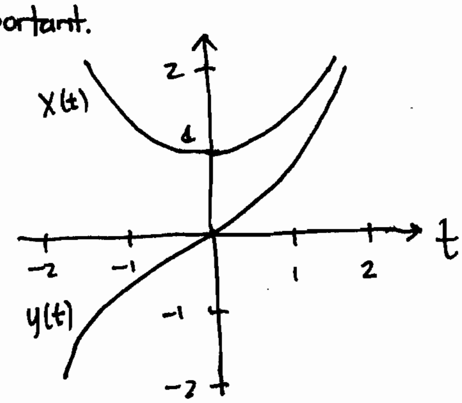
$$\vec{r}\,'(t) = \langle y(t), x(t)\rangle$$

so the tangent to the solution curve equals the value of the vector field $\vec{F}(x,y)=\langle y,x\rangle$ at each point along that curve.

Thus plotting the directionfield for this vector field gives us a picture of the family of solution curves which connect up the arrows.



Of course we are also interested in separate plots of x or y versus t in applications where the individual behavior of $x(t)$ and $y(t)$ is usually important.



Remark: $x^2-y^2=\cosh^2 t-\sinh^2 t = \left(\frac{e^t+e^{-t}}{2}\right)^2-\left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{e^{2t}+2+e^{-2t}-(e^{2t}-2+e^{-2t})}{4}$
$= 1$!

This particular solution is the hyperbolic analog of the usual angle parametrized unit circle, namely a hyperbola. Hyperbolic geometry turns out to be important in special relativity.