Visualizing the initial value problem (cute but not necessary)

IVP: $\begin{cases} \text{2nd order DE for } y(x) \to \text{gen soln } y = c_1 y_1 + c_2 y_2 \\ y(x_0) = y_0 \\ y'(x_0) = v_0 \end{cases}$

$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) = y_0$

$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) = v_0$

Introduce initial data vectors:

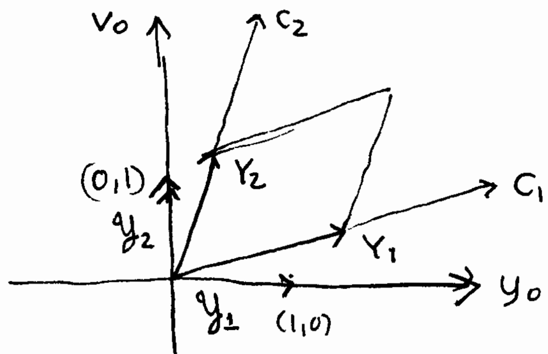
$Y = \begin{bmatrix} y \\ y' \end{bmatrix} \quad Y_1 = \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} \quad Y_2 = \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$

$Y(x_0) = c_1 Y_1(x_0) + c_2 Y_2(x_0) = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$

$\underbrace{\begin{bmatrix} Y_1(x_0) & Y_2(x_0) \end{bmatrix}}_{W(y_1,y_2)(x_0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ v_0 \end{bmatrix}$

Solving the initial conditions is equivalent to trying to express the target initial data vector in terms of a linear combination of the initial data vectors of the basis functions $\{y_1, y_2\}$ for the soln space. The matrix of this linear system is their Wronskian matrix at $x = x_0$.

There is a unique correspondence between initial data and solutions — for each pair (y_0, v_0) there is a unique solution, so we can use these parameters on the space of solutions as coordinates just like we can use (c_1, c_2) as coordinates — they are related to each other by the above linear transformation.



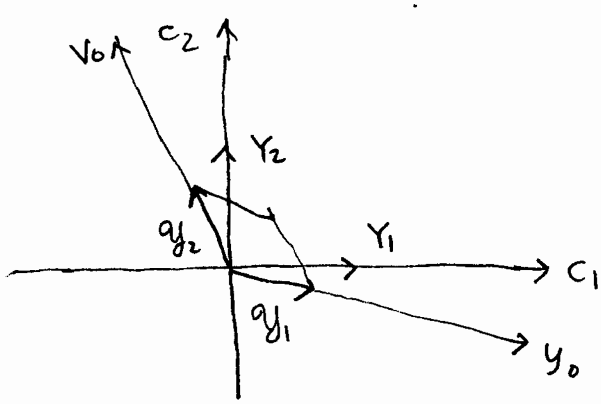
initial data space "=" space of solns

But we dont know in advance what the solution functions (y_1, y_2) are that correspond to $(1,0)$ and $(0,1)$ in the (y_0, v_0) plane.

We start with known (y_1, y_2) from our solution technique.

so rather than thinking of (y_0, v_0) as the old coordinates and (c_1, c_2) as the new coordinates as this picture implies,

we must consider (c_1, c_2) as the old coords & (y_0, v_0) as the new coordinates, as in the second diagram.



space of solutions "=" initial data space

$$\underset{\text{old}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underbrace{W(y_1,y_2)(x_0)^{-1}}_{B} \underset{\text{new}}{\begin{bmatrix} y_0 \\ v_0 \end{bmatrix}}$$

once you backsubstitute these into y, and collect coefficients of y_0, v_0, you find these functions (y_1, y_2):

$y = y_0 y_1 + v_0 y_2$

natural basis for initial conditions at $x = x_0$:

$W(y_1, y_2)(x_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(the columns of B are the coefficients of y_1, y_2 necessary to express the new basis y_1, y_2)

Visualizing the initial value problem (2): example (cute but not necessary)

IVP: $2y'' - 7y' + 3y = 0$, $y(0) = 3$, $y'(0) = 4$

$$\left[\begin{array}{l} y = e^{rx} \nearrow\searrow \quad 2r^2 - 7r + 3 = 0 \to r = \dfrac{7 \pm \sqrt{49 - 4(2)(3)}}{2} = \dfrac{7 \pm 5}{2} = \tfrac{1}{2}, 3 \\ y = e^{\frac{1}{2}x}, e^{3x} = y_1, y_2 \to \text{gen solution}: \quad y = c_1 e^{1/2 x} + c_2 e^{3x} \end{array}\right]$$

Solution technique

ICs: $y = c_1 e^{\frac{1}{2}x} + c_2 e^{3x}$ $\qquad y(0) = c_1 + c_2 = 3$

$y' = \frac{1}{2} c_1 e^{\frac{1}{2}x} + 3 c_2 e^{3x}$ $\qquad y'(0) = \frac{1}{2} c_1 + 3 c_2 = 4$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 3 \end{bmatrix}}_{W(y_1,y_2)(0)} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$W(y_1, y_2) = \begin{bmatrix} e^{x/2} & e^{3x} \\ \frac{1}{2} e^{x/2} & 3e^{3x} \end{bmatrix} \qquad W(y_1,y_2)(0) = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & 3 \end{bmatrix} \longrightarrow W(y_1,y_2)(0)$$

initial data vectors Y_1, Y_2 $\qquad$ $Y_1(0)$ $\quad$ $Y_2(0)$

solving the ICs is equivalent to trying to express $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ as a linear combination of the initial data vectors $Y_1(0), Y_2(0)$ of y_1, y_2.

ICs solution: (mult by inverse)
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 3 & -1 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{6}{5} & -\frac{2}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}}_{B} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} \frac{6}{5}(3) - \frac{2}{5}(4) \\ -\frac{3}{5} + \frac{2}{5}(4) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

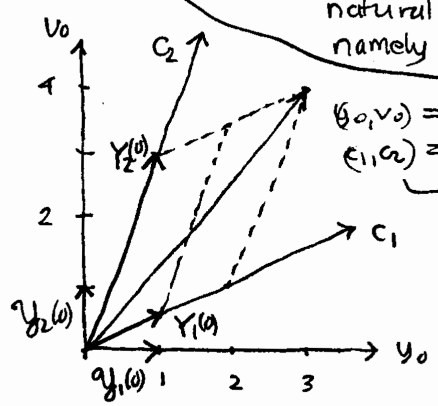
IVP solution: $\boxed{y = 2e^{x/2} + e^{3x}.}$

In general for any initial data:
$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} y_0 \\ v_0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6y_0 - 2v_0 \\ -y_0 + 2v_0 \end{bmatrix}$$

$$y = \tfrac{1}{5}(6y_0 - 2v_0) e^{x/2} + \tfrac{1}{5}(-y_0 + 2v_0) e^{3x} = y_0 \underbrace{\left(\tfrac{6}{5} e^{\frac{x}{2}} - \tfrac{1}{5} e^{3x}\right)}_{y_1} + v_0 \underbrace{\left(-\tfrac{2}{5} e^{x/2} + \tfrac{2}{5} e^{3x}\right)}_{y_2}$$

you can check that y_1, y_2 are the natural basis for initial conditions at $x=0$ namely $W(y_1,y_2)(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(c_1, c_2) coords of y_1, y_2 are columns of B (basis changing matrix)

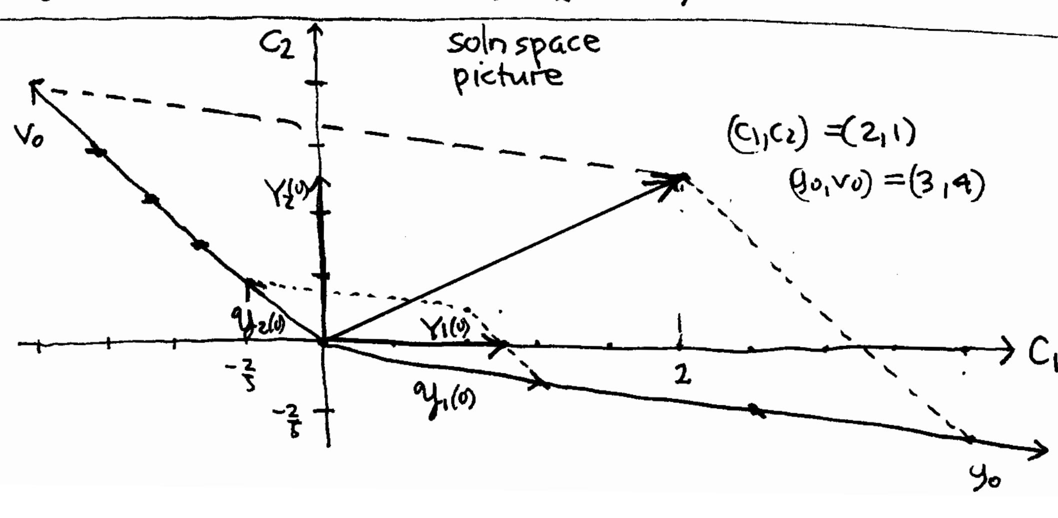


$(y_0, v_0) = (3,4)$
$(c_1, c_2) = (2,1)$
$\longrightarrow \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 2\, Y_1(0) + 1\, Y_2(0)$

initial data space picture (compare diagram with handout on nonstandard coords on $\mathbb{R}^2$)

initial data of y_1, y_2 at $x=0$ provide new basis, with associated coordinates (c_1, c_2).

soln space picture



$(c_1, c_2) = (2,1)$
$(y_0, v_0) = (3,4)$

we usually start with (c_1, c_2) space from solution technique.