Solving Linear Systems Example Revisited

What statements does the solution of $A\vec{x}=\vec{b}$ make about the column vectors $\vec{b}$ and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_7 \in \mathbb{R}^5$ which are the columns of A?

[Note: here we don't know the entries of these vectors since we start with the fully reduced augmented matrix]

$\vec{v}_1 \quad \vec{v}_3 \quad \vec{v}_6 \quad \vec{v}_7$ ← leading 1 cols of A (LI subset) [B vars]

rank(A) = 4 (# lin. ind. cols)

$$
\mathrm{rref}([A,\vec{b}]) = \begin{array}{c} \\ \\ \end{array}
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\
\end{array}
$$

$$
\mathrm{rref}([A,\vec{b}]) = \left[\begin{array}{ccccccc|c}
1 & -2 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 3 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$\vec{v}_2 \quad \vec{v}_4 \quad \vec{v}_5$ ← nonleading 1 cols of A (redundant) [F vars]

- solution:

$$
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2t_1 - t_2 + 2 \\ t_1 \\ -3t_2+3 \\ t_2 \\ t_3 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 2t_1 + (-1)t_2 + 0t_3 + 2 \\ 1t_1 + 0t_2 + 0t_3 + 0 \\ 0t_1 - 3t_2 + 0t_3 + 3 \\ 0t_1 + 1t_2 + 0t_3 + 0 \\ 0t_1 + 0t_2 + 1t_3 + 0 \\ 0t_1 + 0t_2 + 0t_3 + 0 \\ 0t_1 + 0t_2 + 0t_3 + 6 \end{bmatrix} = \underbrace{t_1 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{c}_1} + t_2 \underbrace{\begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{c}_2} + t_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{c}_3}} + \underbrace{\begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \\ 6 \end{bmatrix}}_{\vec{x}_{part}}
$$

if set $=\vec{\emptyset}$, forces $t_1=t_2=t_3=0$

$\vec{x}_{hom} = t_1\vec{c}_1 + t_2\vec{c}_2 + t_3\vec{c}_3$

$A\vec{x}_{hom} = \vec{\emptyset}$

$A\vec{x}_{part} = \vec{b}$

nullspace(A) = span$\underbrace{\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}}$ → LI since $t_1\vec{c}_1 + t_2\vec{c}_2 + t_3\vec{c}_3 = \vec{\emptyset}$ forces $t_1=t_2=t_3=0$

- homogeneous solution:

$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_7\vec{v}_7 = \vec{\emptyset}$

t_1: $2\vec{v}_1 + \vec{v}_2 = \vec{\emptyset} \quad \rightarrow \quad \vec{v}_2 = -2\vec{v}_1$

t_2: $-\vec{v}_1 - \vec{v}_3 + \vec{v}_4 = \vec{\emptyset} \quad \rightarrow \quad \vec{v}_4 = \vec{v}_1 + \vec{v}_3$

t_3: $\vec{v}_5 = \vec{\emptyset} \quad \rightarrow \quad \vec{v}_5 = \vec{\emptyset}$

same for cols of B: each nonleading 1 col is a linear comb of the leading 1 cols to its left with coefficients which are its own entries

3 independent relationships among 7 vectors.

To fix choice of a LI subset, use each one to solve for the last vector in terms of the remaining ones & eliminate from set as redundant:

get rid of: $\vec{v}_2, \vec{v}_4, \vec{v}_5$ (nonleading 1 cols)

keep: $\vec{v}_1, \vec{v}_3, \vec{v}_6, \vec{v}_7$ (leading 1 cols), 4 = rank(A)

- general solution:

$\vec{b} = x_1\vec{v}_1 + \cdots + x_7\vec{v}_7 = (2t_1 - t_2 + 2)\vec{v}_1 + t_1\vec{v}_2 + (-3t_2+3)\vec{v}_3 + t_2\vec{v}_4 + t_3\vec{v}_5 + 0\vec{v}_6 + 6\vec{v}_7$

$= 2\vec{v}_1 + 3\vec{v}_3 + 0\vec{v}_6 + 6\vec{v}_7$ } unique linear comb of LI subset

$+ t_1(2\vec{v}_1 + \vec{v}_2)$
$+ t_2(-\vec{v}_1 - \vec{v}_3 + \vec{v}_4)$
$+ t_3\vec{v}_5$ } ambiguity due to adding multiples of $\vec{\emptyset}$

leading and free variables and columns

$$\underset{m\times n}{A} = \begin{bmatrix} R_1 \\ \vdots \\ R_m \end{bmatrix} = [C_1 \cdots C_n]$$

A matrix can be looked at as an ordered set of rows or as an ordered set of columns.

$$\boxed{A\vec{X} = \vec{0}} \Leftrightarrow \begin{bmatrix} \vec{R}_1 \cdot \vec{x} \\ \vdots \\ \vec{R}_m \cdot \vec{X} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The rows here are the coefficient vectors of a set of linear relationships satisfied by the scalar variables.

two complementary views of the same matrix eqn!

The reduced row echelon form of these equations allows the leading scalar variables to be expressed as linear functions of the free variables, which in turn can take any real values, represented by assigning them to arbitrary constants (parameters).

$$x_1 \vec{C}_1 + \cdots + x_n \vec{C}_n = \vec{0}$$

This represents a linear relationship among the columns if the coeffient vector $\vec{x} \neq \vec{0}$. This relationship only involves those columns whose coefficient in this relationship is nonzero.

The solution of the linear system of equations represents the most general coefficient vector associated with linear relationships among the columns (excluding the zero soln!).

It consists of an arbitrary linear combination of a set of coefficient vectors which is automatically a linearly independent set since if $\vec{x} = t_1 \vec{v}_1 + \cdots + t_p \vec{v}_p = \vec{0}$ the free variable parameters $t_1, \ldots, t_p$ are forced to be zero.

Each coefficient vector $\vec{V}_i$ represents a linear relationship which allows one free column to be expressed as a linear combination of the leading columns.

complementarity:
the leading scalar variables can be expressed as linear combinations of the free variables

BUT

the free columns can be expressed as linear combinations of the leading columns.

Leading & free are exchanged going from variables to columns.

The reduced row echelon form algorithm allows us to see both of these facts by inspection (just looking at the reduced matrix and its equivalent system of equations).

> The free variables can be considered as independent, while the set of leading columns is linearly independent.