③ Trig identities and relative phase of sinusoidal functions

Any linear combination of cosines and sines of the same frequency $\omega > 0$ can be rewritten as a "phase-shifted cosine" using the cosine addition identity:

$$y = \underbrace{\overset{\text{amplitude of } y}{A}\ \cos(\overset{\text{frequency}}{\omega} t - \overset{\text{phase-shift}}{\delta})}_{\text{phase-shifted cosine}} \quad (\omega t - \delta = \text{"phase" of cosine})$$

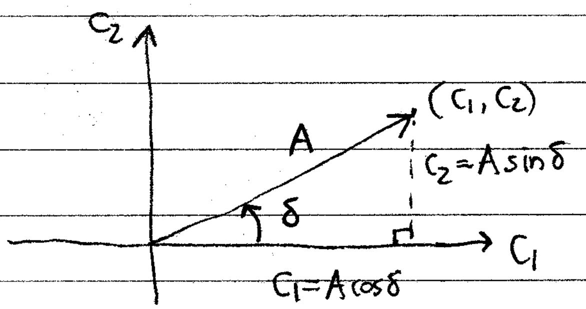
[with nonlinear parameters A and δ]

$$= \underbrace{(A\cos\delta)}_{\equiv c_1} \cos\omega t + \underbrace{(A\sin\delta)}_{\equiv c_2} \sin\omega t$$

$$= \underbrace{c_1 \cos\omega t + c_2 \sin\omega t}_{\text{linear combination of } \cos\omega t \text{ and } \sin\omega t}$$

[with linear parameters c_1 and c_2]

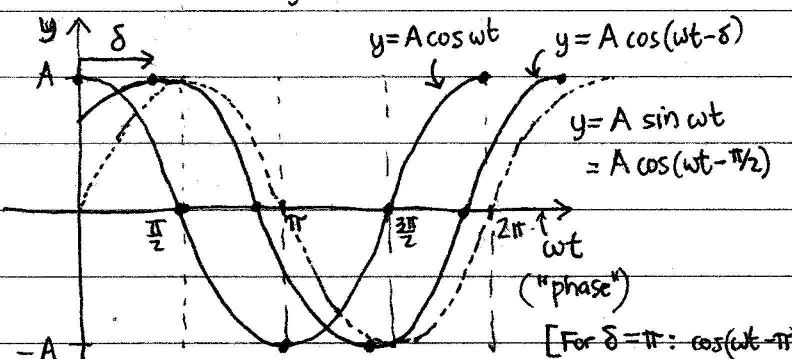
Therefore
$$c_1 = A\cos\delta$$
$$c_2 = A\sin\delta$$
is the relationship between the two different parametrizations of the same 2-parameter family of "sinusoidal functions of frequency ω". This is the exact same relationship which exists between Cartesian coordinates (x, y) in the x-y plane and polar coordinates (r, θ). Thus one finds the amplitude and phase shift of a sinusoidal function given in the linear form [(c_1, c_2) specified] by finding the equivalent polar coordinates (A, δ) of the point (c_1, c_2) in the "linear parameter space", each point of which represents a sinusoidal function.



The inverse relationship is

$$A = \sqrt{(c_1)^2 + (c_2)^2}\ , \qquad \tan\delta = \frac{c_2}{c_1}.$$

The phase shift δ is usually chosen uniquely from the interval $-\pi < \delta \le \pi$.
[See next sheet for how to do this using the inverse tangent.]



For $0 < \delta < \pi$, the phase-shifted cosine "lags" the cosine (its peaks occur after the cosine peaks — or at larger t values).

For $-\pi < \delta < 0$, the phase-shifted cosine "leads" the cosine (its peaks occur before the cosine peaks — or a smaller t values).

[For $\delta = \pi$: $\cos(\omega t - \pi) = -\cos\omega t$, said to be 180° out of phase]

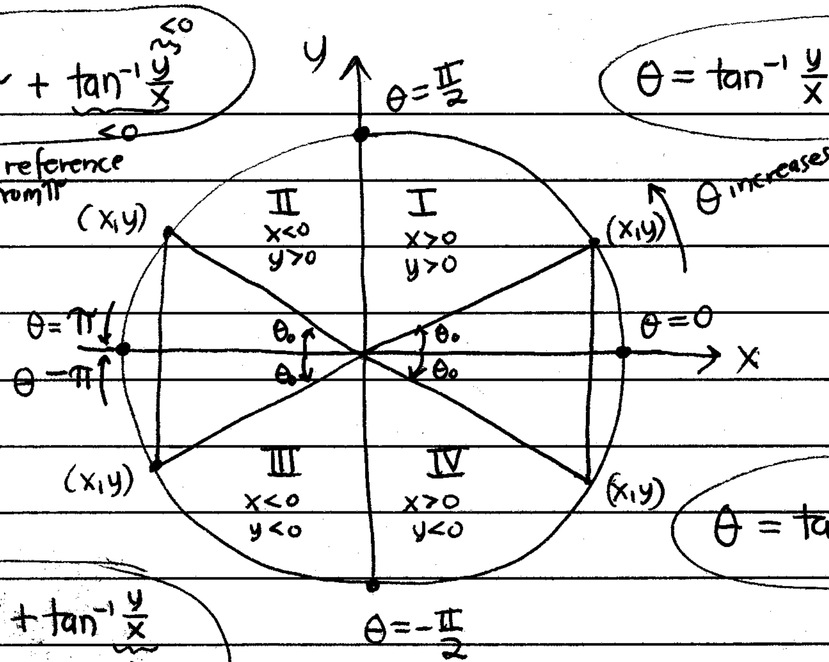
EX. The function $A\sin\omega t = A\cos(\omega t - \pi/2)$ lags the cosine by 90° in phase, or by $\Delta t = \frac{\pi}{2}/\omega$ in t (since $\omega\Delta t = \pi/2$). What are the amplitude and phase shift of $y = \cos\omega t + \sqrt{3}\sin\omega t$? What is its phase-shifted cosine form?

4 ENOUGH TRIGONOMETRY TO FIND THE CORRECT POLAR ANGLE $\in (-\pi, \pi]$

$$\theta = \pi + \underbrace{\tan^{-1}\overset{<0}{\frac{y}{x}}}_{<0}$$

subtract reference angle θ_0 from π

$$\theta = \tan^{-1}\frac{y}{x} > 0$$

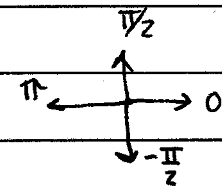


$$\theta = \tan^{-1}\frac{y}{x} < 0$$

$$\theta = -\pi + \underbrace{\tan^{-1}\overset{>0}{\frac{y}{x}}}_{>0}$$

add reference angle θ_0 to $-\pi$

On the 4 axis directions we pick θ to be:



~~In the interior of the 4 quadrants:~~

On the right half of the plane ~~(1st & 4th quadrants)~~

$$\theta = \tan^{-1}\frac{y}{x} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Since the inverse tangent by definition only produces angles in this range, it can't work in the left half of the plane.
~~[The tangent function is not invertible on the whole real line — it is restricted~~ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where it is inverted!]

2nd Quad: Subtract reference angle from π, but

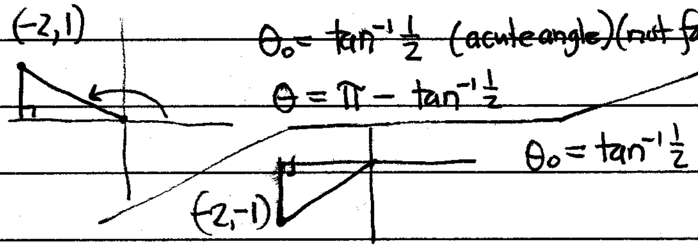
$\theta_0 = \tan^{-1}\left|\frac{y}{x}\right| = -\tan^{-1}\frac{y}{x}$ here so $\qquad \theta = \pi + \tan^{-1}\frac{y}{x} \in \left(\frac{\pi}{2}, \pi\right)$

3rd Quad ~~Add reference angle to π:~~

~~$\theta_0 = \tan^{-1}\left|\frac{y}{x}\right| = \tan^{-1}\frac{y}{x}$ here so~~ $\qquad$ ~~$\theta = -\pi + \tan^{-1}\frac{y}{x} \in \left(-\pi, -\frac{\pi}{2}\right)$.~~

EX. (-2,1) $\quad$ ~~$\theta_0 = \tan^{-1}\frac{1}{2}$ (acute angle)(not familiar one)~~

$\theta = \pi - \tan^{-1}\frac{1}{2}$



$\theta_0 = \tan^{-1}\frac{1}{2}$, $\theta = -\pi + \tan^{-1}\frac{1}{2}$.