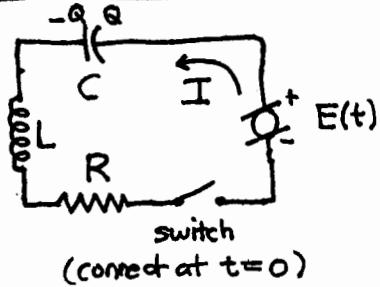


RLC circuits



R resistance (ohms)
L inductance (henries)
C capacitance (farads)

Q charge on capacitor plates (coulombs)
I current (amperes) = $\frac{dQ}{dt}$
E applied voltage (volts) - given

sum of voltage drops
= applied voltage:

$$L \frac{dI}{dt} + RI + \frac{Q}{C} = E(t) \quad (\text{but 2 dependent variables } I, Q)$$

$$I = \frac{dQ}{dt} \rightarrow L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q = E(t) \leftarrow \text{D.E. for } Q$$

but easier to measure current than charge so:

$$\frac{d}{dt} \left(L \frac{dI}{dt} + RI + \frac{Q}{C} \right) = E'(t) \rightarrow \boxed{L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = E'(t)} = 0 \quad \begin{array}{l} \text{if } E(t) = E_0 \\ \text{(DC voltage source)} \\ \text{or if } E_0 = 0 \\ \text{(no battery)} \end{array}$$

$$\downarrow$$

$$L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C} I = 0$$

initial conditions*: no current flow until switch connected at $t=0$ so $I(0)=0$,
but at $t=0$, D.E. says:

$$\underbrace{L I'(0)}_{=0} + \underbrace{R I(0)}_{=0} + \underbrace{\frac{1}{C} Q(0)}_{=0} = E(0) = E_0 \rightarrow I'(0) = \frac{E_0}{L}$$

example: underdamped case, roots: $r = -k \pm i\omega$

$$I = e^{-kt} (c_1 \cos \omega t + c_2 \sin \omega t) \quad I(0) = c_1 \xrightarrow{=} 0$$

$$I' = -k e^{-kt} (c_1 \cos \omega t + c_2 \sin \omega t) + e^{-kt} (-c_1 \omega \sin \omega t + c_2 \omega \cos \omega t) \quad I'(0) = -k c_1 + c_2 \omega = \frac{E_0}{L} \rightarrow c_2 = \frac{E_0}{\omega L}$$

$$\text{so } I = \frac{E_0}{\omega L} e^{-kt} \sin \omega t$$

$$\text{initial amplitude of current } A_0 = \frac{E_0}{\omega L}$$

$k \rightarrow \tau = 1/k$ characteristic decay time

$\omega \rightarrow \pi = \frac{2\pi}{\tau}$ quasi-period

$$\sin(\omega\tau) \rightarrow \frac{\tau}{\pi} = \frac{\tau}{2\pi/\omega} = \frac{\omega\tau}{2\pi} = \# \text{ cycles of oscillation during 1 decay time}$$

$$e^{-k\pi} \rightarrow k\pi = \frac{\pi}{\tau} = \# \text{ factors of } e \text{ by which amplitude decays during 1 oscillation}$$

↑
this ratio or its reciprocal determines whether the oscillation decays slowly or quickly

* subtle point: initially capacitor is uncharged and the current flow is zero. Then the battery is connected and the inductor voltage drop must balance the battery voltage, causing dI/dt to become nonzero and start current flowing

RLC circuit: actual numbers

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = 0 \rightarrow r^2 + R_0r + \omega_0^2 = 0$$

$$\hookrightarrow r = -R \pm i\omega \text{ (underdamped)}$$

$$R = 100 \text{ ohms}$$

$$L = .2 \text{ henries} \quad \left. \right\} \rightarrow R_0 = \frac{R}{L} = \frac{100}{.2} = 500 \text{ sec}^{-1} \rightarrow \tau_0 = R_0^{-1} = \frac{1}{500} = .002 \text{ sec} = 2 \text{ millisecond}$$

$$E_0 = 20 \text{ volts} \rightarrow I'(0) = \frac{E_0}{L} = \frac{20}{.2} = 100 \text{ amps/sec} \quad (\text{when throw switch at } t=0)$$

We fix these and consider various capacitance values which varies the natural frequency $\omega_0 = \sqrt{LC} = \sqrt{.2C}$, and the actual frequency $\omega = \omega_0 \sqrt{1 - \frac{1}{(2R\omega_0)^2}}$.

The decay time for the critically damped / underdamped cases is:

$$k = \frac{R_0}{2} \rightarrow \tau = k^{-1} = 2R_0^{-1} = 2\tau_0 = 4 \text{ millisecond.}$$

- pure damping: $C = \infty, C^{-1} = 0$ (no capacitor) $\rightarrow [\omega_0 = 0, \text{ no oscillations}$
characteristic decay time $T_0 = 2 \text{ millisecond}$

- critical damping: $C = C^{\text{crit}} \Leftrightarrow \frac{1}{2} = \omega_0^{\text{crit}} \tau_0 \rightarrow$

$$\omega_0^{\text{crit}} = \frac{1}{2T_0} = \frac{1}{4 \times 10^{-3}} = 250 \frac{\text{rad}}{\text{sec}} = \frac{250}{2\pi} \text{ Hz} = 39.8 \text{ Hz} \rightarrow 40 \text{ Hz} \text{ (cycles/sec)}$$

$$T_0^{\text{crit}} = 2\pi/\omega_0^{\text{crit}} = .025 \text{ sec} = 25 \text{ millisecond}$$

$$\omega_0^{\text{crit}} = 1/\sqrt{LC^{\text{crit}}} \rightarrow C^{\text{crit}} = 1/(L(\omega_0^{\text{crit}})^2) = \frac{1}{.2(250)^2} = 8 \times 10^{-5} \text{ farad} = 80 \mu\text{farad}$$

$\uparrow 10^{-6} \text{ "micro"}$

- slightly underdamped: $C = \frac{1}{4}C^{\text{crit}} = 20 \mu\text{farad}$

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{(.2)(20 \times 10^{-6})} = \frac{1}{2} \times 10^3 = 500 \frac{\text{rad}}{\text{sec}} = \frac{500}{2\pi} \text{ Hz} = 79.6 \text{ Hz} \rightarrow 80 \text{ Hz.}$$

$$T_0 = 2\pi/\omega_0 = 2\pi/500 = .0125 \text{ sec} = 12.5 \text{ millisecond}$$

$$\omega_0 \tau_0 = \frac{500}{500} = 1 > \frac{1}{2} \text{ (twice critical phase value)}$$

$$\omega = \omega_0 \sqrt{1 - \frac{1}{(2R\omega_0)^2}} = \omega_0 \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2} \omega_0 = 433.0 \frac{\text{rad}}{\text{sec}} = 68.9 \text{ Hz} \rightarrow 69 \text{ Hz}$$

$$T = 2\pi/\omega = \frac{2}{\sqrt{3}} \pi T_0 = 14.5 \text{ millisecond} \quad \hookrightarrow A_0 = \frac{100}{433} = .231 = 231 \text{ milliamp}$$

$$\omega T = \frac{\sqrt{3}}{2} (500)(.0125) = 1.73 \text{ rad} = .275 \text{ cycles of oscillation during 1 decay time}$$

$$e^{-\pi/T} = e^{-14.5/4} = e^{-3.625} = .0267 \sim 2.7\% \text{ left after one oscillation}$$

- very underdamped: $C = C^{\text{crit}}/8000 = 10^{-8} \text{ farad} = .01 \mu\text{farad}$ $\downarrow 10^3 \text{ "Kilo"}$

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{.2 \cdot 10^{-8}} = 22360.7 \frac{\text{rad}}{\text{sec}} = 3560 \text{ Hz} = 3.6 \text{ KHz}$$

$$T_0 = 2\pi/\omega_0 = 2\pi/22360 = 1.76 \text{ millisecond}$$

$$\omega_0 \tau_0 = \frac{22360}{500} = 44.7 \text{ rad} = 7.1 \text{ cycles (of natural oscillation during 1 natural decay time)}$$

$$\omega = \omega_0 \sqrt{1 - 1/44.7^2} = 22359.3 \approx \omega_0 = 3560 \text{ Hz} \quad \rightarrow A_0 = \frac{100}{22360} = .00447 = 4.5 \text{ milliamp}$$

$$\omega T = \frac{3560}{500} = 89.4 \text{ rad} = 14.2 \text{ cycles of actual oscillation during 1 decay time}$$